Lectures on Challenging Mathematics

Calculus 1

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"Cogito ergo Sum" – "I think, therefore I am"

René Descartes (1596–1650)

"Success is not final, failure is not fatal, it is the courage to continue that counts."

Winston Churchill (1874–1965)

"I can see that without being excited, mathematics can look pointless and cold. The beauty of mathematics only shows itself to more patient followers."

Maryam Mirzakhani (1977–2017)

Contents

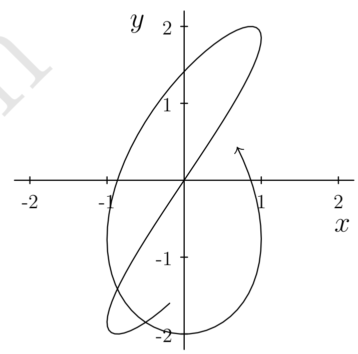
5.4 Derivatives, integrations, and graphs (part 2)

1. Show that $y = x^{\frac{1}{3}}$ has an inflection point at $(0, 0)$. What is unusual here?

2. Use the established result, $\lim_{x\to 0} \frac{\sin x}{x} = 1$, together with judicious substitution, to evaluate each of the following limits.

   (a) $\displaystyle\lim_{x\to 0} \frac{\sin(3x)}{\pi x}$; (b) $\displaystyle\lim_{x\to 0} \frac{\sin(3x)}{\sin(\pi x)}$; (c) $\displaystyle\lim_{x\to \frac{1}{2}} \frac{\cos(\pi x)}{2x-1}$

3. With $f(t) = \sin 4\pi t$ and $g(t) = 2\cos 3\pi t$, verify that the *Lissajous curve* traced by the equations $x = f(t)$ and $y = g(t)$ (shown in part) goes through the origin $(0, 0)$ when $t = 0.5$.

   (a) Even though $\frac{g(0.5)}{f(0.5)}$ makes no sense, $\frac{g(t)}{f(t)}$ does make sense for most other values of t, and the ratio approaches a limiting value m as t approaches 0.5. Find m. What is its significance? It will help to think of $\frac{g(t)}{f(t)}$ as $\frac{g(t)-0}{f(t)-0}$.

   (b) As t approaches 0.5, the ratio of derivatives $\frac{g'(t)}{f'(t)}$ approaches the same limiting value m. Why could this have been expected?

4. *(The first form of) L'Hôpital's Rule.* Let a be a real number and u be a positive real number. Functions f and g are differentiable functions in the (open) interval $(a - u, a + u)$. Suppose that $f(a) = 0 = g(a)$. If $\displaystyle\lim_{t\to a} \frac{g'(t)}{f'(t)}$ exists, then

$$\lim_{t\to a} \frac{g(t)}{f(t)} = \lim_{t\to a} \frac{g'(t)}{f'(t)}.$$

   Explain why *l'Hôpital's Rule* makes sense by considering a curve defined parametrically. (Note that a formal proof is much harder and will be developed shortly.) Identify a few examples of indeterminate forms we worked on recently and apply l'Hôpital's Rule to evaluate them.

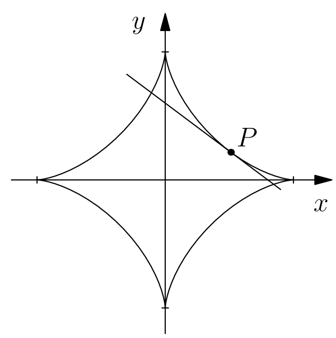
5. Sketch the graph of $x = g(y) = \frac{y^3}{2} + \frac{y}{2} - 1$. Find an equation of the line that is tangent to the graph of $x = g(y)$ at $(-1, 0)$.

6.9 Substitutions and trigonometry in Calculus (part 5)

1. In geometry, an *envelope* of a family of curves in the plane is a curve that is tangent to each member of the family at some point. You can check out the illustrations of this definition at en.wikipedia.org/wiki/Envelope_(mathematics). What is the envelope for the family of segments PQ with P on the x-axis and Q on the y-axis with $PQ = 1$? (You might want to raw segment PQ many times.)

   The graph of this envelope is called *astroid*. It is a type of cycloid, this curve is traced by a point on a wheel that rolls without slipping around the inside of a circle whose radius is four times the radius of the wheel – to be shown in the near future. Leibniz studied the curve in 1715. (Do not mix up *astroid* and *asteroid* which is a small, planet-like member of our solar system.)

2. Suppose that $P = (a, b)$ is a first-quadrant point on the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$. The line tangent to the curve at P crosses the x-axis at Q and the y-axis at R. Show that $Q = \left(a^{\frac{1}{3}}, 0\right)$. Show that the y-coordinate of R, can be expressed in terms of b alone. Show that the length of segment QR does not depend on where P is on the curve. Finally, explain the name of this curve.



3. (Continuation) Find parametric equation of the curve by first rewriting the equation as $\left(x^{\frac{1}{3}}\right)^2 + \left(y^{\frac{1}{3}}\right)^2 = 1$. Express the coordinates for P, Q, R parametrically and determine the length of QR.

4. Make up an example of a differential equation whose isoclines are parallel to the x-axis, and whose slopes vary between -2 and 2, inclusive. Sketch some of the solution curves. If you can, find equations to describe them.

5. Evaluate each of the following.

   (a) $\displaystyle\int_0^1 \frac{1}{x^2+1}\,dx$ (b) $\displaystyle\int_0^1 \frac{1}{x^2+4}\,dx$ (c) $\displaystyle\int_0^1 \frac{1}{x^2+2x+5}\,dx.$

6.13 A short review – practice set 2

1. When asked to find the antiderivative of $p(x) = \sin x \cos x$, Ronaldo got $R_1(x) = \frac{1}{2}\sin^2 x$, Ronaldinho got $R_2(x) = -\frac{1}{2}\cos^2 x$, and CR7 got $R_3(x) = -\frac{1}{4}\cos 2x$. Explain each of their methods. How do functions $R_1(x), R_2(x), R_3(x)$ relate to each other?

2. Find each of the following.

   (a) $\dfrac{d\left(\int_3^x t\sqrt{1+t^2}\,dt\right)}{dx}$

   (b) $D_x\left(\int_4^x \sin t(1+\cos t)^3\,dt\right)$

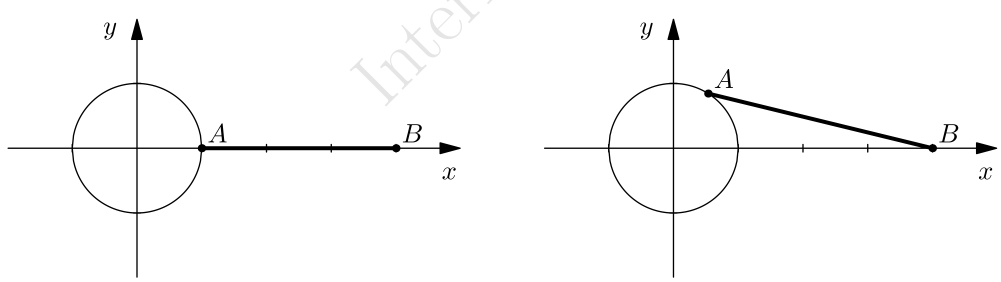
3. Consider the differential equation $\dfrac{dy}{dx} = \dfrac{1-x}{y}$ with $y > 0$. Let $y = f(x)$ be a particular solution to the given differential equation with the initial condition $f(4) = 4$.

   (a) Use Euler's method, starting at $x = 4$ with *two* steps of equal size, to approximate $f(5)$.

   (b) Find an expression of this particular solution $y = f(x)$. (To check if your answers make sense, the exactly value of $f(5)$ and the estimation value of $f(5)$ should be reasonably close to each other.)

   (c) For some constant c, line $y = c$ is tangent to the graph of $y = f(x)$. Find the coordinates of the tangent point. Determine whether f has a local maximum, local minimum, or neither at this point.

4. A wheel of radius 1 is centered at the origin, and a rod AB of length 3 is attached at A to the rim of the wheel. The wheel turns in a counterclockwise direction, one rotation every 2π seconds, and, as it turns, the other end $B = (x, 0)$ of the rod is constrained to slide back and forth along a segment of the x-axis. The top figure shows this apparatus when $t = 0$, and $t = 1.02$ produces the bottom figure. Verify that, for any time t, the position of B is given by $x = \cos t + \sqrt{9 - \sin^2 t}$. When is B moving faster — when A is at the top of the wheel or when the rod AB is tangent to the wheel? Calculate $\dfrac{dx}{dt}$ to find out.



5. For n an integer, evaluate

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2-0^2}} + \frac{1}{\sqrt{n^2-1^2}} + \frac{1}{\sqrt{n^2-2^2}} + \cdots + \frac{1}{\sqrt{n^2-(n-1)^2}}\right).$$

6.20 Some abstract concepts of Calculus (part 5)

1. Convince a skeptic that $\int_a^b kf(x)\,dx$ is equivalent to $k\int_a^b f(x)\,dx$ when k is a constant.

2. Kelly completed a 250-mile drive in exactly 5 hours—an average speed of 50 mph. The trip was not actually made at a constant speed of 50 mph, of course, for there were traffic lights, slow-moving trucks in the way, etc.

   (a) Nevertheless, there must have been at least one instant during the trip when Kelly's speedometer showed exactly 50 mph. Give two explanations—one using a distance-versus-time graph, and the other using a speed-versus-time graph. Make your graphs consistent with each other!

   (b) A student drew the line that joins $(0, 0)$ to $(5, 250)$, and remarked that any actual distance-versus-time graph must have points that lie above this line *and* points that lie below it. What do you think of this remark and why?

   (c) Another student thought that the area between the distance-versus-time graph and the time axis was a significant number. Explain what you think of this idea.

3. Find $D_x\left(\int_4^{x^2} \frac{\sin t}{t}\,dt\right)$.

4. The function f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Consider the following statement:

   If f is increasing on $[a, b]$, then $f' \geq 0$ in the interval (a, b).

   (a) Prove this statement is true.

   (b) What is the converse of this statement? Show that the converse statement is also true.

   (c) What is analogous statement for a decreasing function?

5. *The Mean Value Theorem for Integrals* says: If f is a function that is continuous for $a \leq x \leq b$, then there is a number c between a and b for which $f(c) \cdot (b - a) = \int_a^b f(x)\,dx$. Interpret this statement. Then, by applying the Fundamental Theorem of Calculus, show that the equation is actually a consequence of the Mean Value Theorem for Derivatives.