# Lectures on Challenging Mathematics

## XC 7 part 3

## Advanced Precalculus and Introduction to Calculus

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*"Cogito ergo Sum" – "I think, therefore I am"*

René Descartes (1596–1650)

*"Success is not final, failure is not fatal, it is the courage to continue that counts."*

Winston Churchill (1874–1965)

*"I can see that without being excited, mathematics can look pointless and cold. The beauty of mathematics only shows itself to more patient followers."*

Maryam Mirzakhani (1977–2017)

# Contents



**1 More on conic curves and matrices** **3**
1.1 The inverse of a matrix (part 1) . . . . . . 3
1.2 Hyperbola (part 1) . . . . . . 5
1.3 Hyperbola (part 2) . . . . . . 6
1.4 The inverse of a matrix (part 2) . . . . . . 8
1.5 Hyperbola (part 3) . . . . . . 10
1.6 Revisiting the compositions of transformations and the products of matrices . . . . . 11
1.7 Hyperbola (part 4) . . . . . . 12
1.8 Markov process and recursion (part 1) . . . . . . 13
1.9 Hyperbola (part 5) . . . . . . 15
1.10 Markov process and recursion (part 2) . . . . . . 16

**2 A short review with extensions** **17**
2.1 Sinusoidal functions and their graphs (part 1) . . . . . . 17
2.2 Interest and growth rates (part 1) . . . . . . 19
2.3 Sinusoidal functions and their graphs (part 2) . . . . . . 20
2.4 Interest and growth rates (part 2) . . . . . . 21
2.5 Sinusoidal functions and their graphs (part 3) . . . . . . 22
2.6 Instantaneous rate of change (part 1) . . . . . . 23
2.7 Introduction to functional properties (part 1) . . . . . . 24
2.8 Instantaneous rate of change (part 2) . . . . . . 25
2.9 Introduction to functional properties (part 2) . . . . . . 26
2.10 Instantaneous rate of change (part 3) . . . . . . 27
2.11 Practices with functional properties (part 1) . . . . . . 28
2.12 Revisiting sinusoidal functions and their graphs . . . . . . 29
2.13 Practices with functional properties (part 2) . . . . . . 31
2.14 A short review of Instantaneous rate of change . . . . . . 32
2.15 Practices with functional properties (part 3) . . . . . . 33

**3 Limits and approximations** **35**
3.1 Limits and approximations (part 1) . . . . . . 35
3.2 Algebraic knowledge and tools (part 1) . . . . . . 37

3.3 Limits and approximations (part 2) . . . . . 38
3.4 Limits and approximations (part 3) . . . . . 39
3.5 Algebraic knowledge and tools (part 2) . . . . . 40
3.6 Limits and approximations (part 4) . . . . . 41
3.7 Limits and approximations (part 5) . . . . . 42
3.8 Algebraic knowledge and tools (part 3) . . . . . 44
3.9 Limits and approximations (part 6) . . . . . 45
3.10 Limits and approximations (part 7) . . . . . 47
3.11 Algebraic knowledge and tools (part 4) . . . . . 48
3.12 Limits and approximations (part 8) . . . . . 49
3.13 Limits and approximations (part 9) . . . . . 51
3.14 Algebraic knowledge and tools (part 5) . . . . . 53
3.15 Limits and approximations (part 10) . . . . . 54

**4 Projects for XC7 part 3** **55**
4.1 Challenges in 3-D geometry (part 1) . . . . . 55
4.2 Vector projection and linear regression (part 1) . . . . . 56
4.3 Computations with exponents and logarithms (part 2) . . . . . 58
4.4 Challenges in 3-D geometry (part 2) . . . . . 59
4.5 Vector projection and linear regression (part 2) . . . . . 60

**5 Supplementary materials** **61**
5.1 Cross product . . . . . 61
5.2 Repaying loans . . . . . 62
5.3 Revisiting addition and subtraction formulas . . . . . 64

## 1.9 Hyperbola (part 5)

1. Let $F_1 = (12, 0)$ and $F_2 = (-12, 0)$. All the points $P$ such that $|PF_1 - PF_2| = 18$ form a curve.
   (a) What is the name of that curve?
   (b) What are the vertices of the curve?
   (c) Sketch an accurate diagram of the curve. In particular, how does the graph look like from a great distance?
   (d) Find an equation of the curve. (You *do not* need to derive it.)
   (e) Find the equations of the asymptote lines of the curve.
   (f) Find a parametric equation of the curve.
   (g) Find an equation for each of the directrix of the curve.

2. (Continuation) Find an equation of the line that is tangent to the graph at $A(-12, a)$ by two approaches. One method uses approximation, the other applies reflection property. (If you need more practice, find an equation of the line that is tangent to the graph at $B(72, b)$.)

3. A hyperbola equation can be written in *factored form*, as in $(4x + 3y)(4x - 3y) = 72$. This enables the *asymptotes* to be written down easily: $4x + 3y = 0$ and $4x - 3y = 0$. What is the reasoning behind this statement? Apply this reasoning to sketch the following graphs (all actually hyperbolas). Draw the asymptotes first, then plot one convenient point (an axis intercept, for example), then use symmetry to freehand the rest of a rough sketch.
   (a) $(x + 2y)(x - 2y) = 36$ (b) $(x + 2y)(x - 2y) = -36$ (c) $xy = 18$ (Hmm ...)

4. The graph of $xy = 18$ looks like a hyperbola. Assuming it is a hyperbola, what could be its asymptotes, vertices, focal points, eccentricity, and directrix.

5. (Continuation) Show that the graph of $xy = 18$ is a hyperbola by following any one of the two definitions of a hyperbola.

## 1.10 Markov process and recursion (part 2)

1. Let

$$\mathbf{M} = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}.$$

Draw the vectors $\mathbf{v}, \mathbf{M}\mathbf{v}, \mathbf{M}^2\mathbf{v}, \mathbf{M}^3\mathbf{v}, \ldots$ with their tails at the origin. Explain why the heads define a sequence of collinear points. Find the property of matrix $\mathbf{M}$ that is responsible for the collinearity.

2. (Continuation) Show that the distances between consecutive points form a geometric sequence. What is the limiting position of this sequence of points?

3. (Continuation) Does this limiting position depend on the choice of $\mathbf{v}$? If so, in what way? Does this limiting position depend on the choice of $\mathbf{M}$ (assuming that $\mathbf{M}$ is a meaningful transition matrix)? If so, in what way?

4. The graph of $xy = 4$ looks like a hyperbola.

   (a) Assuming it is a hyperbola. We can rotate it around the origin in the *clockwise* direction to obtain a hyperbola in standard position (with foci lying on the $x$-axis). What is the size of the rotation angle?

   (b) Let $P = (x, y)$ be a general point on the graph of $xy = 4$. Let $Q$ be the image of $P$ under the rotation. Find transformation matrices $\mathbf{M}$ and $\mathbf{N}$ such that

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \mathbf{M} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{N} \begin{bmatrix} X \\ Y \end{bmatrix}.$$

   (c) What is the relation between matrices $\mathbf{M}$ and $\mathbf{N}$?

5. (Continuation) One of the above transformation matrices will help you more directly to find an equation for point $Q = (X, Y)$. Which one? Confirm that the graph of $xy = 4$ is a hyperbola.

## 2.10 Instantaneous rate of change (part 3)

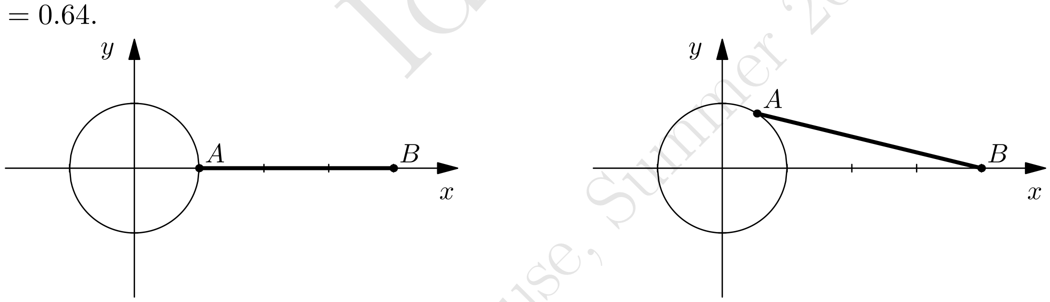
1. Let $P = (x, \sin x)$ be a point on the sine graph (in radian mode). Explain why $\dfrac{\sin x}{x}$ can be interpreted as the slope of a special line passing though $P$. As $x$ approaching 0, what will be the *limiting* position of this special line?

2. Consider the sine curve in radian mode. Confirm that The line $y = 0.5x$ intersects the sine curve in three places by finding these intersection points. Consider all straight lines of *positive* slope that can be drawn through the origin. If one of these lines is randomly chosen, what is the probability that it will intersect the sine graph more than once?

   What would happen if we consider the sine curve in degree mode?

3. Your calculator probably cannot graph the equation $\sin(x + y) = \frac{1}{2}$ as it is written, but you should be able to draw the graph yourself. Try it, use radian mode.

4. A wheel of radius 1 meter is centered at the origin, and a rod $AB$ of length 3 meters is attached at $A$ to the rim of the wheel. The wheel is turning counterclockwise, one rotation every 4 seconds, and, as it turns, the other end $B = (b, 0)$ of the rod is constrained to slide back and forth along a segment of the $x$-axis. Given any time $t$ seconds, the position of $B$ is determined. This functional relationship is expressed by writing $b = f(t)$. The figure on the left shows this apparatus when $t = 0$ and $b = 4$, and the figure on the right corresponds to $t = 0.64$.




   (a) Calculate $f(0)$, $f(1)$, $f(2)$, $f(271)$, $f(314)$.

   (b) Explain why $f$ is a *periodic* function of $t$, and sketch a graph of $b = f(t)$ for $0 \leq t \leq 8$. What is the range of values of $f$?

   (c) Calculate a general formula for $f(t)$, and use it to refine your sketch.

   (d) In degree mode, write an equation for a sinusoidal curve that matches the all the crests and valleys of the graph of $y = f(t)$.

5. (Continuation) What is the velocity of $B$ when $t = 1$? Find an instant $t$ of time when you think $B$ is moving most rapidly. Explain your choice.

## 3.14 Algebraic knowledge and tools (part 5)

1. Let $a$ be a positive real number. When asked to evaluate $\lim\limits_{n\to\infty} n^a$ and $\lim\limits_{n\to\infty} a^n$, Alex wrote

$$\lim_{n\to\infty} n^a = \left(\lim_{n\to\infty} n\right)^a = \infty^a = \infty \quad \text{and} \quad \lim_{n\to\infty} a^n = \left(\lim_{n\to\infty} a\right)^n = a^n$$

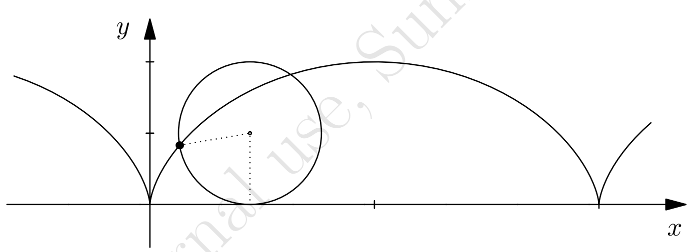
   Help Alex to identify the errors.

2. For positive integers $k$ and $n$ with $2 \le k \le n$, show that $\dfrac{\binom{n}{k}}{n^k} < \dfrac{\binom{n+1}{k}}{(n+1)^k}$.

   *Hint.* It might be useful to notice that $\dfrac{n+1}{n} < \dfrac{n}{n-1} < \cdots < \dfrac{n-(k-2)}{n-(k-1)}$.

3. Let $n$ be a positive integer $n$ and $x$ be a positive real number less than 1.
   (a) Find a quick way to show that $(1+x)^n \ge 1 + nx$.
   (b) If $0 < x < 1$, the inequality $(1-x)^n \ge 1 - nx$ is a bit harder to prove. Verify that it is true for $n = 1, 2, 3$. Show that if the statement $(1-x)^n \ge 1-nx$ is true for integer $n = k$ (starting with $n = 1, 2, 3$), the statement is also true for the next integer $n = k+1$. This means that the statement is true for all positive integers. Explain this reasoning process. It is called the principle of *mathematical induction*.

4. A wheel of radius 1 rolls along the $x$-axis without slipping. A mark on the rim follows a *cycloid* that starts at origin $O$. Find both coordinates of the mark after the wheel rolls a distance $t$, where $t < \frac{\pi}{2}$. Check your formulas to see whether they are also correct for $\frac{\pi}{2} \le t$.




   Input your expressions for both coordinates into a graphing device to obtain an accurate graph of the cycloid for $0 \le t \le 6\pi$. Compare this graph with your own sketch.

   You must work in the *parametric equation* mode when you operate your graphing device. Furthermore, it is necessary to work in the radian mode. Why?

5. Show that for any positive integer $n > 2$, we have

$$\frac{\binom{n}{2}}{n^2} + \frac{\binom{n}{3}}{n^3} + \cdots + \frac{\binom{n}{n}}{n^n} < 1.$$