Lectures on Challenging Mathematics

XC 7 Part 1

Analytic geometry, vectors, and 3-D geometry

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"Cogito ergo Sum" – "I think, therefore I am"

René Descartes (1596–1650)

"Success is not final, failure is not fatal, it is the courage to continue that counts."

Winston Churchill (1874–1965)

"I can see that without being excited, mathematics can look pointless and cold. The beauty of mathematics only shows itself to more patient followers."

Maryam Mirzakhani (1977–2017)

Contents

1 Analytic geometry and vectors **2**
1.1 Vector motion (part 1) 2
1.2 Linear parametric equations (part 1) 4
1.3 Lattice points (part 1) 5
1.4 Vector motion (part 2) 6
1.5 Linear parametric equations (part 2) 7
1.6 Lattice points (part 2) 8
1.7 Vector motion (part 3) 9
1.8 Linear parametric equations (part 3) 10
1.9 3-D rectangular coordinates 11
1.10 Vector motion (part 4) 12
1.11 Linear parametric equations (part 4) 13
1.12 Vector motion (part 5) 14
1.13 Analytic transformations (part 1) 15
1.14 Lattice points (part 3) 16
1.15 Analytic transformations (part 2) 17
1.16 3-D rectangular coordinates and linear equations (part 1) 18
1.17 Analytic transformations (part 3) 19
1.18 Linear parametric equations (part 5) 20
1.19 3-D rectangular coordinates and linear equations (part 2) 21
1.20 Linear parametric equations (part 6) 22
1.21 Analytic transformations (part 4) 23
1.22 Vector motion (part 6) 24
1.23 Linear parametric equations (part 7) 25
1.24 Analytic transformations (part 5) 26
1.25 Lattice points (part 4) 27

2 Solid geometry and vector operations **28**
2.1 Prisms 28
2.2 Pyramids 30
2.3 Cones (part 1) 31
2.4 Scaling in 3-D geometry 32

2.5 Cones (part 2) . . . 33
2.6 Vector operations in Euclidean geometry (part 1) . . . 34
2.7 Sphere . . . 35
2.8 Vector operations in Euclidean geometry (part 2) . . . 36
2.9 Volume of a sphere . . . 37
2.10 Polar coordinates . . . 38
2.11 Vector form of the law of cosines (part 1) . . . 39
2.12 Trigonometric and geometric calculations (part 1) . . . 40
2.13 Vector projection (part 1) . . . 41
2.14 Computations with cones, prisms, and spheres (part 1) . . . 42
2.15 Vector form of the law of cosines (part 2) . . . 43
2.16 Kepler's Third Law and planet geometry . . . 44
2.17 Vector projection (part 2) . . . 45
2.18 Trigonometric and geometric calculations (part 2) . . . 46
2.19 Revisiting parametric equations and vector motions . . . 47
2.20 Computations with cones, prisms, and spheres (part 2) . . . 48

3 Additional practices for XC7 Part 1 **49**
3.1 Practices with vectors and analytic geometry (part 1) . . . 49
3.2 Practices with vectors and analytic geometry (part 2) . . . 51
3.3 Practices with vectors and analytic geometry (part 3) . . . 52
3.4 Practices with vectors and analytic geometry (part 4) . . . 53
3.5 Practices with vectors and analytic geometry (part 5) . . . 54

1.23 Linear parametric equations (part 7)

1. A photon, a particle which transmits light, is traveling in the coordinate plane according to the equation $P_t = (50-7t, 60-8t)$. A mirror is placed along the line $x = 1$. Once the photon hits the mirror, the particle's path gets reflected.

   (a) Find the location of the particle at $t = 10$.

   (b) Find a parametric equation of the motion of the particle after reflection.

2. The position of an airplane that is approaching its airport is described parametrically by

$$P_t = (100, 50, 90) + t[-100, -50, -90].$$

   For what value of t is the airplane closest to the traffic control center located at $(34, 68, 16)$? (Solve this problem first by finding two perpendicular vectors and then by minimizing a quadratic function.)

3. Let $A_1 = (3, -1)$, $A_2 = (23, 11)$, $A_3 = (11, 31)$, $A_4 = (-9, 19)$, $P = (13, 5)$, and $Q = (20, 16)$.

   (a) Explain why there is a transformation that sends A_1, A_2, A_3, A_4 to A_1, A_4, A_3, A_2, respectively. What are the images of P and Q under this transformation?

   (b) Explain why there is a transformation that sends A_1, A_2, A_3, A_4 to A_3, A_2, A_1, A_4, respectively. What are the images of P and Q under this transformation?

4. Find the image of the point (m, n) after it is reflected across the line $ax + by = c$. Find the distance from point (m, n) to line $ax + by = c$.

5. Find the distance between the lines $ax + by = c_1$ and $ax + by = c_2$.

1.24 Analytic transformations (part 5)

1. Point by point, the transformation $\mathcal{T}(x, y) = (4x - y, 3x - 2y)$ sends the line $x + 2y = 6$ onto an image line. Find an equation of this image line.

2. A *similarity transformation* is a geometric transformation that uniformly multiplies distances, in the following sense: For some positive number m, and *any* two points A and B and their respective images A' and B', the distance from A' to B' is m times the distance from A to B.

   (a) Is it true that the transformation $\mathcal{T}(x, y) = (3x, 2y)$ is a similarity transformation? Explain.

   (b) Show that any dilation transforms any figure into a similar figure.

   (c) Determine if the dilation $\mathcal{T}(x, y) = (mx, my)$ is a similarity transformation.

   (d) Given two similar figures, it might not be possible to transform one into the other using only a dilation. Explain this remark.

3. A *quarter-turn* is a 90-degree rotation. If the counterclockwise quarter-turn centered at $(3, 2)$ is applied to $(7, 1)$, what are the coordinates of the image? What are the image coordinates when this transformation is applied to a general point (x, y)?

4. A dilation $\mathcal{T}$ sends $A = (2, 3)$ to $A' = (5, 4)$, and it sends $B = (3, -1)$ to $B' = (7, -4)$. Where does it send $C = (4, 1)$? Write a general formula for $\mathcal{T}(x, y)$.

5. Consider the transformation $\mathcal{T}(x, y) = \left(\dfrac{\sqrt{3}x + y}{2}, \dfrac{x - \sqrt{3}y}{2}\right)$.

   (a) Show that it is an isometry.

   (b) Apply the transformation to a triangle whose vertices are $O = (0, 0)$, $A = (4, 0)$, and $B = (0, 2)$. (Why do we choose these points?) How do you convince your peers that $\mathcal{T}$ does not represent a rotation?

   (c) Consider an arbitrary point in the plane $C = (m, n)$. Explain why the angles in triangle COC' is not fixed, where C' denotes the image of C.

   (d) Find a way to convince your peers that $\mathcal{T}$ represents a reflection.

   Query: It is possible to rewrite $\mathcal{T}$ in a different form to reveal its geometric nature. How?

2.19 Revisiting parametric equations and vector motions

1. Draw and compare the vectors

$$[x, y], \quad [y, x], \quad 0.6[x, -y] + 0.8[y, x].$$

2. In trapezoid $ABCD$, $AB \parallel CD$ and $AB/CD = 2/3$. Diagonals AC and BD meet at P. Express the following in terms of $\mathbf{u} = \overrightarrow{DA}$ and $\mathbf{v} = \overrightarrow{DC}$.
   (a) $\overrightarrow{AB}$ (b) $\overrightarrow{CB}$ (c) $\overrightarrow{AC}$ (d) $\overrightarrow{DP}$ (e) $\overrightarrow{BP}$

3. Let $ABCDEFGH$ be a rectangular box, with $ABCD$ and $EFGH$ being two of its faces and AE and BF being two its edges. Suppose that $A = (0, 0, 0)$, $B = (4, 0, 0)$, $D = (0, 3, 0)$, and $E = (0, 0, 2)$. The midpoint of GH is M.
   (a) Find coordinates for M.
   (b) Find the coordinates for point P on segment AC that is 2 units from A.
   (c) Decide whether angle APM is a right angle, and give your reasons.
   (d) Victor wants find the point on segment AC that is closest to M, and his solution is $\frac{\sqrt{17}}{25}(4, 3, 0)$. We have studied multiple approaches to find this point. What is your favorite approach? Based on your approach, explain without actual computation, why Victor's solution is wrong. What is the approach likely used by Victor? Find this point by your second favorite approach.

4. Penta chooses five of the vertices of a unit cube. What is the maximum possible volume of the figure whose vertices are the five chosen points?

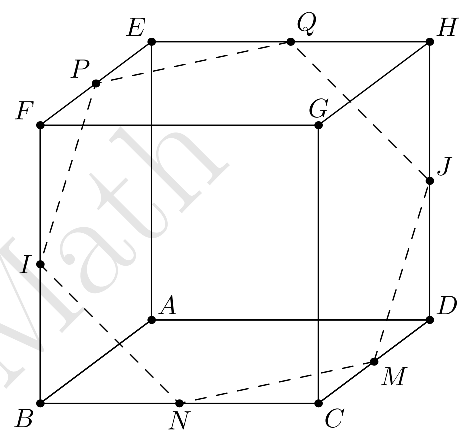
5. A car travels due east at $\frac{2}{3}$ mile per minute on a long, straight road. At the same time, a circular storm, whose radius is 51 miles, moves southeast at $\frac{\sqrt{2}}{2}$ mile per minute. At time $t = 0$, the center of the storm is 110 miles due north of the car. At time $t = t_1$ minutes, the car enters the storm circle, and at time $t = t_2$ minutes, the car leaves the storm circle. Find $t_1 + t_2$.

2.20 Computations with cones, prisms, and spheres (part 2)

1. Assume that as a cubical bar of soap is used, all edges shrink at a constant rate of n units per day. Starting with a full bar, the soap was used for 6 days and its surface area was cut in half. Starting with a full bar of soap, compute exactly the time it would take for the volume to become one-eighth of the original volume.

2. The figure at right shows a $2 \times 2 \times 2$ cube $ABCDEFGH$, as well as respective midpoints M, N, P, Q of edges DC, CB, FE, EH. It so happens that M, N, I, P, Q, J all lie in a plane. Can you justify this statement? Describe hexagon $MNIPQJ$ and find its area.

   Query. Is it possible to obtain a polygon with a larger area by slicing the cube with a different plane? If so, show how to do it. If not, explain why it is not possible.



3. A right circular cylinder with its diameter equal to its height is inscribed in a right circular cone. The cone has diameter 10 and altitude 12, and the axes of the cylinder and cone coincide. Find the radius of the cylinder.

4. Sphere $\mathcal{S}_1$ is inscribed in a regular tetrahedron $\mathcal{T}$. Sphere $\mathcal{S}_2$ is circumscribed about $\mathcal{T}$. Find the ratio of the volume of $\mathcal{S}_1$ to that of $\mathcal{S}_2$.

5. Let $ABCD$ be a regular tetrahedron. Four regular tetrahedrons $ABCX$, $BCDY$, $CDAZ$, and $DABW$ are erected outside of $ABCD$ (one per face). If the volume of $ABCD$ is 1, what is volume of $XYZW$?