Lectures on Challenging Mathematics

Calculus 2

Summer 2021

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"Cogito ergo Sum" – "I think, therefore I am"

René Descartes (1596–1650)

"Success is not final, failure is not fatal, it is the courage to continue that counts."

Winston Churchill (1874–1965)

"I can see that without being excited, mathematics can look pointless and cold. The beauty of mathematics only shows itself to more patient followers."

Maryam Mirzakhani (1977–2017)

Contents

4.14 More general techniques and concepts in Calculus (part 4)

1. Find each of the following.

   (a) $\displaystyle\lim_{x\to 0} \frac{x^4}{x^2 - x + 2\cos x}$ (b) $\displaystyle\int_0^{\ln 2} \tanh t \, dt$ (c) $D_x\left((\sin x)^{\cos x}\right)$

2. We generalize the concept of the centroid of a region to the "centroid" of a curve—roughly speaking—the average of the points on the curve.

   (a) What is the centroid of a line segment? What is the centroid of the four line segments forming the boundary of a parallelogram? What is the centroid of an ellipse (just the boundary, not including its interior)?

   (b) Explain why the centroid of an circular arc is *not* located at the midpoint of the arc.

3. Given an arc, its centroid is—roughly speaking—the average of the points on the arc. For example, consider the semicircular arc $y = \sqrt{1 - x^2}$ for $-1 \le x \le 1$. The centroid is usually denoted by coordinates $(\overline{x}, \overline{y})$. Because this arc is symmetric about the y-axis, it is clear that $\overline{x} = 0$. Thus only $\overline{y}$ needs to be calculated. As usual, it is convenient to approximate the arc by a series of inscribed segments, whose lengths are $\Delta L = \sqrt{\Delta x_i^2 + \Delta y_i^2}$. Imagine that M points have been distributed uniformly along all of these segments. The task confronting you is to add all of their y-coordinates and then divide by M.

   (a) For any one of these segments, it is reasonable to use the *same* y-value for every point found on the segment. What y-value would you use, and why is it reasonable?

   (b) If y_i represents a segment whose length is ΔL, then this segment has about $\dfrac{\Delta L}{\pi} \cdot M$ of the points, and it therefore contributes $y_i \cdot \dfrac{\Delta L}{\pi} \cdot M$ to the sum of all the y-coordinates. Explain this reasoning.

   (c) The average of the y-coordinates of all the points on these segments is approximately
   $$\sigma = y_1\frac{\sqrt{\Delta x_1^2 + \Delta y_1^2}}{\pi} + y_2\frac{\sqrt{\Delta x_2^2 + \Delta y_2^2}}{\pi} + y_3\frac{\sqrt{\Delta x_3^2 + \Delta y_3^2}}{\pi} + \cdots + y_n\frac{\sqrt{\Delta x_n^2 + \Delta y_n^2}}{\pi}.$$
   It is not difficult to see that σ can be viewed as a Riemann sum and this sum approaches $\displaystyle\frac{1}{\pi}\int_{-1}^{1} y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ as the number of segments grows. Explain these remarks.

   (d) Complete the calculation of $\overline{y}$.

4. What is the relation between the solution curves for the differential equation $\dfrac{dy}{dx} = \dfrac{y}{2}$ and the solution curves for the differential equation $\dfrac{dy}{dx} = \dfrac{y}{2}$? Find equations for the two curves (one from each family) that pass through the point $(0, 2)$.

5. Evaluate $\displaystyle\int_0^{\infty} x^{100} e^{-x}\, dx$ and $\displaystyle\int_0^{1} (1 - x^2)^{100}\, dx$.

5.15 Convergence and Taylor's Theorem (part 4)

1. Use Maclaurin series (instead of l'Hôpital's Rule) to evaluate $\lim_{t\to 0} \dfrac{t\cos t - \sin t}{t - \sin t}$.

2. Find a series $\sum a_n$ satisfying the following conditions:

   (i) One can apply the Root Test to show it is convergent.

   (ii) One could not draw a conclusion about convergence by applying the Ratio Test.

   Query: Can you find such a series if the roles of the Ratio Test and Root Test are swapped in the conditions?

3. Determine with justification if the following statement is true.

   If the series $\sum_{n=1}^{\infty} a_n$ converges, then the series $\sum_{n=1}^{\infty} a_n^2$ converges.

4. Determine with justification if the following statement is true

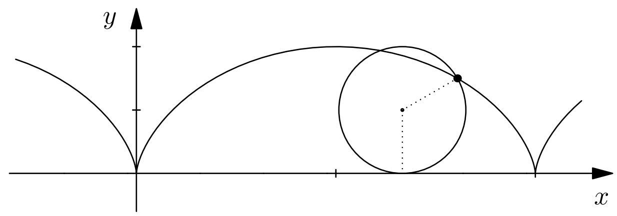
   If the series $\sum_{n=1}^{\infty} a_n$, with $a_n \geq 0$, converges, then the series $\sum_{n=1}^{\infty} a_n^2$ converges.

5. Determine with justification if the following statement is true.

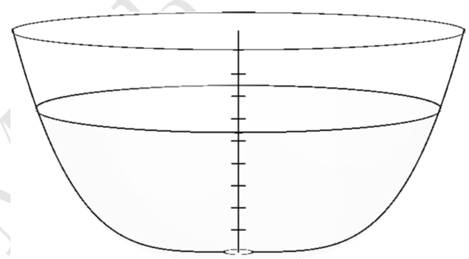
   If the series $\sum_{n=1}^{\infty} a_n$ converges, then the series $\sum_{n=1}^{\infty} a_n^3$ converges.

5.16 Review and extension set 8

1. We have shown that one arch of the cycloid $(x, y) = (t - \sin t, 1 - \cos t)$ is exactly 8 units long. Find coordinates for the point on this curve that is 6 units from the origin, the distance being measured along the curve.



2. In ancient times, a *clepsydra* was a bowl of water that was used to time speeches. As the water trickled out through a small hole in the bottom of the bowl, time was measured by watching the falling water level. Consider the clepsydra that is obtained by revolving the curve $y = x^4$ for $0 \le x \le 1$ around the y-axis. Use Torricelli's Law to show that the water level in this bowl will drop *at a constant rate*.



According to current records, the oldest Surviving Water Clock or Clepsydra was made between 1417 to 1379 BCE. Interested readed mgiht want to visit https://www.historyofinformation.com/detail.php?id=3067

3. A *torus* (a mathematical term used to describe the surface of a bagel) is obtained by revolving a circle of radius a around an axis that is b units from the circle's center. Without doing any integration, provide an intuitive value for the area of this surface, and explain your thinking. Then confirm your answer by setting up and evaluating a suitable integral.

4. Evaluate $\displaystyle\lim_{n\to\infty} \frac{2}{n}\left(\sqrt{\frac{2}{n}} + \sqrt{\frac{4}{n}} + \sqrt{\frac{6}{n}} + \cdots + \sqrt{\frac{2n}{n}}\right)$.

5. Find a solution, in closed form, of the differential equation $xy' - y = \dfrac{4x^2}{1+2x}$ satisfying the initial condition $f(1) = 1 + 2\ln 3$, where $y = f(x)$.

5.19 Convergence and Taylor's Theorem (part 5)

1. Consider the sequence $x_n = \left(\frac{1}{2}\right)^n \left|\sin \frac{n\pi}{2}\right| - \frac{1}{n}\left|\cos \frac{n\pi}{2}\right|$ for every positive integer n.

   (a) Write out the first four terms of this sequence, and show that $x_n \to 0$ as $n \to \infty$.

   (b) Is the series $\sum_{n=1}^{\infty} x_n$ alternating? Explain. Does it converge? Explain.

2. Find all real numbers x such that the series $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$ is absolutely convergent.

3. For every positive integer n, let $s_n = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!}\left(\frac{\pi}{3}\right)^{2k}$. Let $s = \lim_{n\to\infty} s_n$.

   (a) Explain why s is known to be between s_5 and $s_5 + \frac{3^{11}}{2^{11}\cdot 11!}$.

   (b) What is the exact difference between s and s_5?

4. For a nonzero series $\sum a_n$, we can illustrate the relationship between the Ratio Test and the Root Test with a simple calculation:

$$|a_n|^{\frac{1}{n}} = (|a_0|)^{\frac{1}{n}} \left(\left|\frac{a_1}{a_0}\right|\right)^{\frac{1}{n}} \left(\left|\frac{a_2}{a_1}\right|\right)^{\frac{1}{n}} \cdots \left(\left|\frac{a_n}{a_{n-1}}\right|\right)^{\frac{1}{n}};$$

   that is, the right-hand side is the limit of the geometric means of the first n consecutive ratios of the series. Conceptually speaking, if these consecutive ratios have a limit (say L), then their geometric mean shall have the same limit L—because we are taking the geometric mean of a large number of terms with almost all them approximately equal to L. To be more concise (but yet *not* a complete proof), complete the following paragraph. (In Analysis, we study the concepts of *limit inferior* and *limit superior* of a sequence. With those concepts, it is not difficult to clear up the following argument formally. In the next chapter, we will also learn better way to prove this fact.)

   Let $(a_1, a_2, \dots)$ be a nonzero sequence with $\lim_{n\to\infty}\left|\frac{a_n}{a_{n-1}}\right| = L$. For arbitrarily ________ $\epsilon > 0$ (with $L > \epsilon$), there is a ________ large positive integer N, such that we have $0 < L - \epsilon < \left|\frac{a_i}{a_{i-1}}\right| < L + \epsilon$ for every ________. For every ________, we can write

$$\underline{\hspace{2cm}} = (|a_N|)^{\frac{1}{n}} \left(\left|\frac{a_{N+1}}{a_N}\right|\right)^{\frac{1}{n}} \cdots \left(\left|\frac{a_n}{a_{n-1}}\right|\right)^{\frac{1}{n}}.$$

   It follows that for every ________, we have

$$(|a_N|)^{\frac{1}{n}} (L-\epsilon)^{\frac{n-N}{n}} < \underline{\hspace{2cm}} < (|a_N|)^{\frac{1}{n}} (L+\epsilon)^{\frac{n-N}{n}}.$$

Because N and a_N are fixed, we also have

$$\lim_{n\to\infty} (|a_N|)^{\frac{1}{n}} = 1, \quad \lim_{n\to\infty} (L+\epsilon)^{\frac{n-N}{n}} = L+\epsilon, \quad \lim_{n\to\infty} (L-\epsilon)^{\frac{n-N}{n}} = L-\epsilon.$$

We see that for large n, $|a_n|^{\frac{1}{n}}$ can be *sandwiched* in between quantities that can be made arbitrarily close to L.

While the Root Test is a stronger then the Ratio Test, the Ratio Test is much more practical in many situations because many series (in particular, Taylor series) involve various forms of $n!$ in their expressions.

Query: Do you know why the argument given above is *not* a complete rigorous proof yet?

5. We have proved the Limit Comparison Test which states

Suppose that $\{a_n\}$ and $\{b_n\}$ are two *positive* sequences, and that $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n}$ is a finite number L. The infinite series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

The question is: Does the statement remain true if the terms in the sequences are *not* always positive? In other words, determine whether the following statement is true.

Suppose that $\{a_n\}$ and $\{b_n\}$ are two nonzero sequences, and that $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n}$ is a finite number L. The infinite series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.