Lectures on Challenging Mathematics

Math Challenges 6

Geometry

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1.7 Area, similarity, and Ceva's theorem (part 3)

1. The arcs of four quarter-circles are drawn inside a circle intersecting the circle in pairs of points: (A, B), (B, C), (C, D), (D, A), respectively. If all of the five circles have radius 1 cm, what is the area, in square centimeters, of the region enclosed by the four quarter-circles?

2. Construct, with only an unmarked straight edge, a triangle RGB (of your choice, certainly it *doesn't have to be equilateral*) with an interior point P such that such that the area ratio of triangles PGB, PBR, PRG is $[PGB] : [PBR] : [PRG] = 2 : 3 : 5$.

   Accurate drawing is the theme of the problem. You should use the *graph paper* well and choose your triangle (not necessarily an equilateral triangle at all) wisely to avoid any unnecessary estimations. You can only draw lines with the straight edge, and you can't estimate any non-lattice points unless they are the intersection of the grid lines or the lines constructed by the straight edge.

3. In triangle RGB, point X divides side RG in the ratio $RX : XG = m : n$, and point Y divides side GB in the ratio $GY : YB = p : q$. Let C be the intersection of segments BX and RY. Find the ratios of areas
   (a) $[CGB] : [CBR]$;
   (b) $[CBR] : [CRG]$;
   (c) $[CGB] : [CRG]$;
   (d) $[CGB] : [CBR] : [CRG]$.

   Also find the ratio into which the line GC divides the side BR.

4. Mixtures of *three* quantities can be modeled geometrically by using a triangle of area 1. For example, if point P is inside the triangle ABC, then $[APB] + [BPC] + [CPA] = 1$.

   (a) What geometric figure would be suitable for describing a mixture of *two* quantities?

   (b) Kirby wants to use a point inside a unit square to describe a mixture of *four* quantities. After some careful thought, he declares that this method does not work and lists a few reasons. Which of these reasons are right?

   - There is no way to assign a triangle a negative quantity.
   - No triangle can represent a quantity that is more than 0.5.
   - Two of the triangles together always represent a total quantity of 0.5.

   (c) What geometric figure would be suitable for describing a mixture of *four* quantities? Provide details on constructing such mixture.

5. In rectangle $ABCD$, $AB = 11$ and $BC = 13$. Points P and Q lie on sides AB and AD respectively with $BP = 2$. Diagonal BD meets segments CP and CQ in X and Y, respectively. Given that the area of triangle CXY is the equal to the sum of the areas of triangles BPX and DQY, find the length of segment DQ.