Lectures on Challenging Mathematics

Math Challenges 6

Counting

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1. Two identical decks of cards are shuffled together. What is the probability that the first 52 cards will contain all eight Aces?

2. Consider a $3 \times 3$ point array. Five of the nine points will be colored in red, and the rest will be colored in blue.

   (a) Determine the total number of coloring schemes. (Two coloring schemes are considered different even though one can be obtained from the other via a rotation and/or a reflection.)

   (b) Some of these coloring schemes satisfy the following property: There exist three blue points that are collinear and there does not exist three red points that are collinear. How many such coloring schemes are there?

3. Find the smallest positive integer $n$ for which the expansion of $(xy - 3x + 7y - 21)^n$, after like terms have been collected, has at least 1996 terms.

4. A 5-tuple $(x_1, x_2, x_3, x_4, x_5)$ of nonnegative integers is called magical if exactly one of the five numbers is odd and the sum of these five numbers is equal to 99. How many magical 5-tuples are there?

5. A license plate consists of 8 digits. It is called even if it contains an even number of 0’s. Find the number of even license plates.