Lectures on Challenging Mathematics

Math Challenges 6

Algebra

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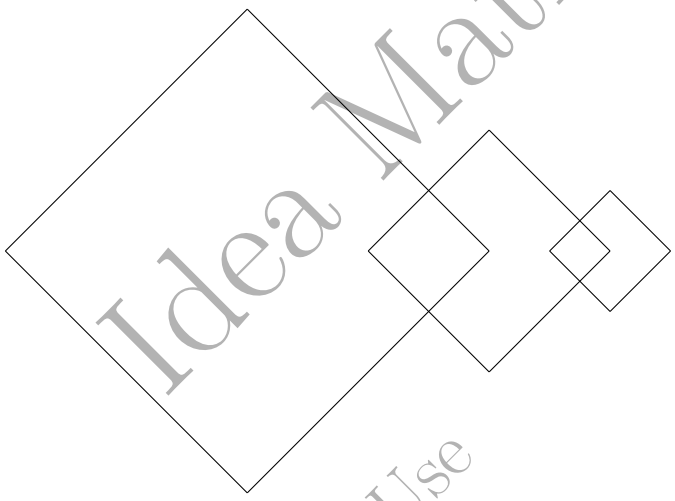
Contents

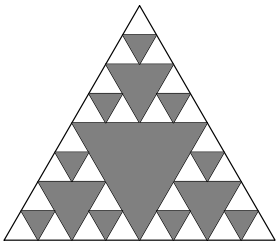
1.5 Fractals and recursive relations (part 1)

1. In the sequence $2001, 2002, 2003, \ldots$, each term after the third is found by subtracting the previous term from the sum of the two terms that precede that term. For example, the fourth term is $2001 + 2002 - 2003 = 2000$. What is the 2004^{th} term in this sequence?

2. Square S_1 is 1×1. For $i \geq 1$, the lengths of the sides of square S_{i+1} are half the lengths of the sides of square S_i, two adjacent sides of square S_i are perpendicular bisectors of two adjacent sides of square S_{i+1}, and the other two sides of square S_{i+1} are the perpendicular bisectors of two adjacent sides of square S_{i+2}. Let $\mathcal{R}$ denote region consisting of points lying in at least one of $S_1, S_2, \ldots, S_{10}$. Find the total area of $\mathcal{R}$.



3. An equilateral triangle of unit area is painted step-by-step as follows: Step 1 consists of painting the triangle formed by joining the midpoints of the sides. Step 2 then consists of applying the same midpoint-triangle process to each of the three small unpainted triangles. Step 3 then consists of applying the midpoint-triangle process to each of the nine very small unpainted triangles. The result is shown at right.



In general, each step consists of applying the midpoint-triangle process to each of the (many) remaining unpainted triangles left by the preceding step. Let P_n be the area that was painted *during* step n, and let U_n be the total unpainted area left after n steps have been completed.

Find U_1, U_2, U_3, P_1, P_2, and P_3. Write a recursive description of U_n in terms of U_{n-1}. Find an explicit formula for U_n.

4. (Continuation) Write a recursive description of P_n in terms of P_{n-1}. Find an explicit formula for P_n.

Use your work to evaluate the sum

$$\frac{1}{4}+\frac{3}{16}+\frac{9}{64}+\cdots+\frac{3^{99}}{4^{100}}+\frac{3^{100}}{4^{101}}.$$

Express the series using sigma notation.

5. Ten guys sit in ten seats in a line. All ten guys get up and then reseat themselves using all ten seats, each sitting in the seat he was in before or a seat next to the one he occupied before. In how many ways can the guys be reseated?