Lectures on Challenging Mathematics

Integrated Mathematics 6

Geometry

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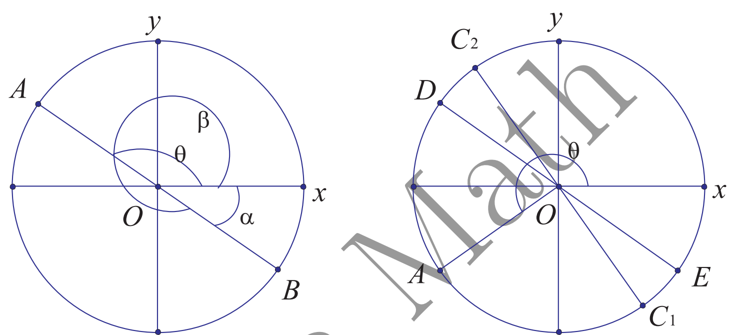
Contents

1.14 Circular parametric equations (part 3)

1. Assume that $A = (\cos\theta, \sin\theta)$. Let B be the point on the unit circle that is diametrically opposite to A. Write the coordinates of B in two ways to express each of

$$\sin(\theta \pm 180^\circ) \quad \text{and} \quad \cos(\theta \pm 180^\circ)$$

in terms of $\sin\theta$ and $\cos\theta$.



2. Similarly, by rotating point A around the origin 90° in the clockwise direction (to point C_2), in the counterclockwise direction (to C_1), reflecting across the x axis (to D), and reflecting across the y axis (to E), express each of the following in terms of $\sin\theta$ and $\cos\theta$:

$$\sin(\theta \pm 90^\circ), \quad \cos(\theta \pm 90^\circ), \quad \sin(-\theta), \quad \cos(-\theta), \quad \sin(180^\circ - \theta), \quad \cos(180^\circ - \theta).$$

3. (Continuation) Furthermore, by either reflecting A across the line $y = x$ or using the second and third formulas above, we can show that $\sin(90^\circ - \theta) = \cos\theta$ and $\cos(90^\circ - \theta) = \sin\theta$. This is the reason behind the nomenclature of the "cosine" function: "cosine" is the complement of sine, because the angles $90^\circ - \theta$ and θ are complementary angles. All these interesting and important trigonometric identities are based on the geometric properties of the unit circle.

   Given a unit circle in the coordinate plane. Explain how to plot the point $(\sin 20^\circ, \cos 40^\circ)$ with a protractor and a straight edge.

4. Consider the set $\{\sin(90 \pm 35^\circ), \cos(90 \pm 35^\circ)\}$. How many elements are in the set? What trigonometric identities did you find while solving this exercise?

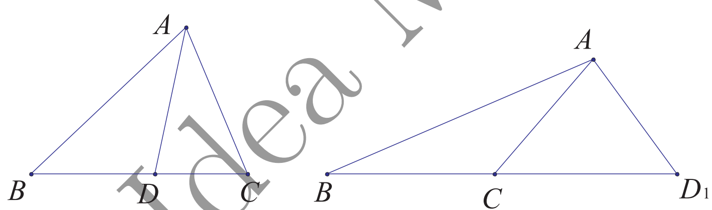
5. The point $A = (\cos\theta, \sin\theta)$ is 3 units away from the point $B = (2\cos 75^\circ, 2\sin 75^\circ)$. If $0^\circ \le \theta < 360^\circ$, compute θ.

1.19 Applications of the laws of sines and cosines (part 2)

1. From the window in Mr. Fat's classroom, it is estimated that the distance to the front door of Phillips Hall is 45 yards. A turn of the head through an angle of 63° counterclockwise enables one to stare directly at the academy mail room door, estimated to be 65 yards away. At that instant, Mr. Aft is spotted leaving the mail room through that very door and walking directly to the front door of Phillips Hall. If he appears to arrive at the door about 48 seconds later, what would be a good estimate of his average speed during the trip? Please explain your approach.

2. [Angle-bisector theorem] The angle-bisector theorem states that:

   Let ABC be a triangle, and let D be a point on segment BC such that $\angle BAD = \angle CAD$. Then
   $$\frac{AB}{AC} = \frac{BD}{CD}.$$



   This theorem can be proved by a synthetic approach. Indeed, we can extend segment BA through A to E such that $CE \parallel AD$ and consider the similar triangles BAD and BEC. We leave these details to the reader as simple exercises.

   Now prove this theorem by using trigonometry.

3. The angle-bisector theorem can be extended to the situation in which AD_1 is the external angle bisector of the triangle. State and prove this result.

4. A parallelogram has a 7-inch side and a 9-inch side, and the longer diagonal is 14 inches long. Find the length of the other diagonal. Do you need your calculator to do it? You should not. Indeed, you should try two approaches without calculator: An approach with the law of cosine and an approach with vector operations.

   This result can be formulated to general result for a parallelogram. One can start with "In a parallelogram, the sum of the squares of lengths of its sides ..." Complete this statement and prove it.

5. In triangle ABC, $(a+b+c)(a+b-c) = 3ab$. Determine the measure of $\angle C$.