Lectures on Challenging Mathematics

Integrated Mathematics 3

Algebra (part 2)

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Zuming Feng
Phillips Exeter Academy and IDEA Math
zfeng@exeter.edu

Contents

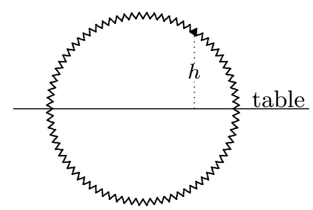
1.6 Graph of the sine function

1. The *angular velocity* of an object is the rate at which it rotates around a chosen center point. One way to describe it is to tell how many degrees per unit of time the object rotates around the center.

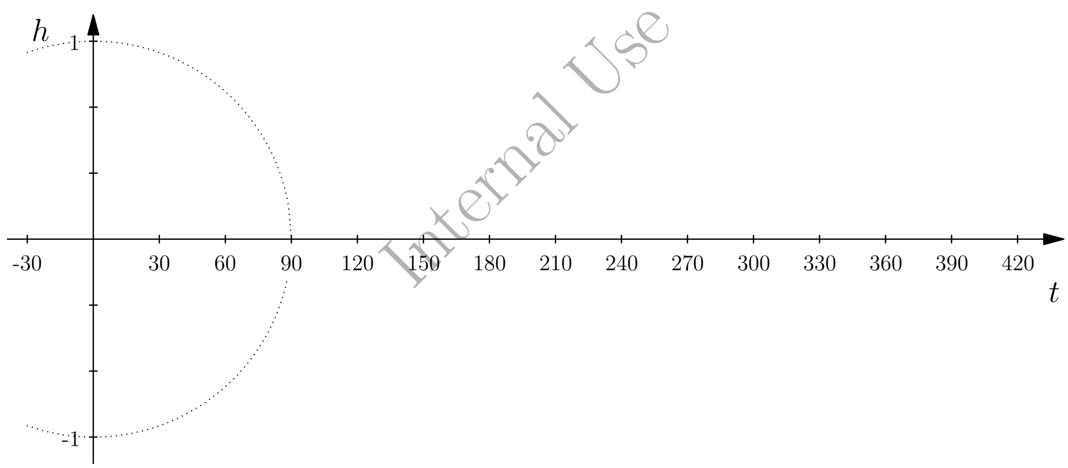
   Think of a clock with a second, minute, and hour hands. Describe hands' angular speed in degrees travelled around the circle in one hour. When is the first time after midnight the minute and hour hands point in the same direction?

2. Andy is riding a merry-go-round, whose radius is 25 feet and which is turning 36 degrees per second. Seeing a friend in the crowd, Andy steps off the outer edge of the merry-go-round and suddenly finds it necessary to run. At how many feet per second?

3. A large circular saw blade with a 1-foot radius is mounted so that exactly half of it shows above the table. It is spinning slowly, at one degree per second counterclockwise. One tooth of the blade has been painted red. This tooth is initially 0 feet above the table, and rising.

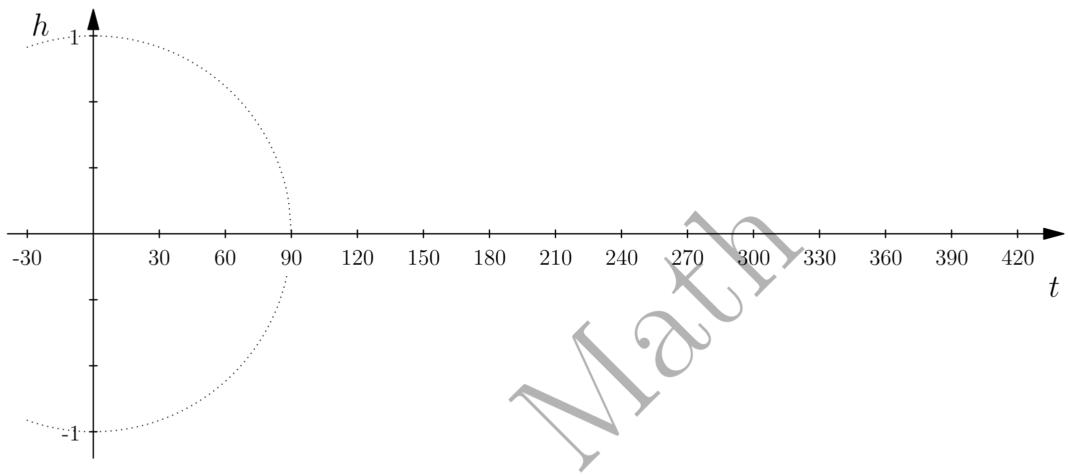


   What is the height of the tooth after 30, 45, 60, and 90 seconds? After t seconds? Draw by hand a graph that shows how the height h of the red tooth is determined by the elapsed time t. The unit circle in the diagram will help you plot the values of h more precisely. It is customary to say that h is a function of t.



4. The graph of the height h of the red saw tooth is an example of a sine curve. Draw $y = \sin x$ on a graphing tool and compare it with the graph that you drew in the preceding exercise. How many times does the graph of the sine curve intersect the x-axis? Does the graph repeat its shape?

5. (Continuation) Suddenly the saw blade starts to rotate twice faster. The tooth at that moment is again 0 feet above the table, and rising. Sketch the new graph of the height h in relation with the elapsed time t. Which trigonometric function have you plotted?



1.17 Fractals and the infinite geometric series

1. Square S_1 is a unit square. We construct a sequence of squares $S_1, S_2, S_3, \ldots$. The lengths of the sides of square S_{i+1} are half the lengths of the sides of square S_i. The sides of the square S_{i+1} are parallel to the sides of S_i and the bottom left vertex of S_{i+1} coincides with the top right vertex of S_i. An example of the first five squares is drawn below:

   (a) Let $\mathcal{R}_n$ denotes the connected region consisting of the first n squares. Find the area and the perimeter of $\mathcal{R}_n$ in terms of n.

   (b) Suppose $\mathcal{R}$ is the region obtained when n goes to infinity. Determine the area and the perimeter of $\mathcal{R}$.

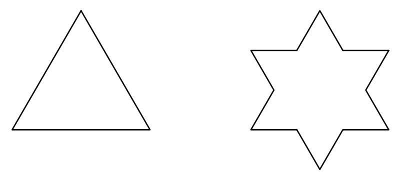
2. When Alicia was finding the area of $\mathcal{R}$, she said that even though the area of the figure increases with every additional square, from the beginning she knew that the area of the figure is finite. Do you agree with her comment?

   Ben, in his turn, said that he knows how to compute the perimeter of $\mathcal{R}$, if its value is finite. Because if P is finite, he said, then $4 + \frac{1}{2}P = P$ and therefore $P = 8$. Can you explain Ben's idea? Use Ben's reasoning to find the area of $\mathcal{R}$.

3. In 1904 Helge von Koch invented his *snowflake*, which is probably the first published example of a *fractal*. Fractals are objects that look similar and preserve a certain pattern at every scale. We are about to construct one!

   The figure we will obtain will be the result of an endless sequence of stages:

   Stage 0 (the initial configuration) consists of an equilateral triangle, whose sides are 1 unit long. Stage 1 is obtained from stage 0 by replacing the *middle third* of each edge by a pair of segments, arranged so that a small equilateral triangle protrudes from that edge. In general, each stage is a polygon that is obtained by applying the middle-third construction to *every* edge of the preceding stage.



Make your own sketch of stages 2 and 3. Then answer the questions below and place your results into the table.

Stage	0	1	2	3	...	n
# edges	3				...	
# vertices	3				...	
Length of the edge	1				...	
# triangles added	0				...	
Perimeter	3				...	
Area					...	

(a) Stage 0 has three edges, and stage 1 has twelve. How many edges do stages 2 and 3 have? How many edges does stage n have?

(b) Stage 1 has twelve vertices. How many vertices does stage n have?

(c) How long is each edge of stage 1? of stage 2? of stage n?

4. (Continuation) What is the perimeter of stage 1? of stage 2? of stage n? Does the snowflake have finite perimeter? Explain.

5. (Continuation) Is the area enclosed by the snowflake finite? Explain. Try to do so without any computation first.

   The area enclosed by stage 0, the initial equilateral triangle, is $\frac{\sqrt{3}}{4}$. What is the area enclosed by stage 1? by stage 2? by stage n? Show that the area enclosed by the completed snowflake can be obtained with the help of a geometric series.