Lectures on Challenging Mathematics

Math Challenges 5

Counting

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1.10 Counting practice (part 3)

1. For a permutation $p = (a_1, a_2, \ldots, a_9)$ of $(1, 2, \ldots, 9)$, let $n(p)$ denote the maximum of the three products $a_1a_2a_3, a_4a_5a_6, a_7a_8a_9$, and let $m$ denote the minimum value of $n(p)$ for all possible permutations $p$. Determine the number of permutations $p$ with $n(p) = m$.

2. There are ten girls and four boys in Mr. Fat’s combinatorics class. In how many ways can these students sit around a circular table such that no boys are next to each other?

3. Suppose that $E = 7^7$, $M = 7$, and $C = 7 \cdot 7 \cdot 7$. Four letters $E, M, C, C$ are arranged randomly in the following blanks.

   ___ × ___ × ___ × ___

   Then one of the multiplication signs is chosen at random and changed to an equals sign. What is the probability that the resulting equation is true?

4. Draw 42 points $P_1, \ldots, P_{42}$ equally spaced around the circumference of a circle. How many (unordered) triples $\{P_i, P_j, P_k\}$ are there so that triangle $P_iP_jP_k$ is obtuse?

5. Let set $S = \{1, 2, 3, 4, 5, 6\}$, and let set $T$ be the set of all subsets of $S$ (including the empty set and $S$ itself). Let $t_1, t_2, t_3$ be elements of $T$, not necessarily distinct. The ordered triple $(t_1, t_2, t_3)$ is called satisfactory if either

   (a) both $t_1 \subseteq t_3$ and $t_2 \subseteq t_3$; or
   (b) both $t_3 \subseteq t_1$ and $t_3 \subseteq t_2$.

   Compute the number of satisfactory ordered triples $(t_1, t_2, t_3)$. 