Lectures on Challenging Mathematics

Math Challenges 5

Counting

Summer 2018

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1.10 Counting practice (part 3)

1. For a permutation $p = (a_1, a_2, \ldots, a_9)$ of $(1, 2, \ldots, 9)$, let $n(p)$ denote the maximum of the three products $a_1a_2a_3, a_4a_5a_6, a_7a_8a_9$, and let m denote the minimum value of $n(p)$ for all possible permutations p. Determine the number of permutations p with $n(p) = m$.

2. There are ten girls and four boys in Mr. Fat's combinatorics class. In how many ways can these students sit around a circular table such that no boys are next to each other?

3. Suppose that $\mathtt{E} = 7^7$, $\mathtt{M} = 7$, and $\mathtt{C} = 7 \cdot 7 \cdot 7$. Four letters E, M, C, C are arranged randomly in the following blanks.

$$\_\_\_ \times \_\_\_ \times \_\_\_ \times \_\_\_$$

Then one of the multiplication signs is chosen at random and changed to an equals sign. What is the probability that the resulting equation is true?

4. Draw 42 points $P_1, \ldots, P_{42}$ equally spaced around the circumference of a circle. How many (unordered) triples $\{P_i, P_j, P_k\}$ are there so that triangle $P_iP_jP_k$ is obtuse?

5. Let set $S = \{1, 2, 3, 4, 5, 6\}$, and let set T be the set of all subsets of S (including the empty set and S itself). Let t_1, t_2, t_3 be elements of T, not necessarily distinct. The ordered triple (t_1, t_2, t_3) is called *satisfactory* if either

   (a) both $t_1 \subseteq t_3$ and $t_2 \subseteq t_3$, or

   (b) both $t_3 \subseteq t_1$ and $t_3 \subseteq t_2$.

   Compute the number of satisfactory ordered triples (t_1, t_2, t_3).