Lectures on Challenging Mathematics

Math Challenges 5

Algebra

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1.8 Recursive relations (part 2)

1. The Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ starts with two 1s, and each term afterwards is the sum of its two predecessors. Which one of the ten digits is the last to appear in the units position of a number in the Fibonacci sequence?

2. Another interesting recursion is at the heart of an interesting unsolved problem known as the *Collatz conjecture*. Consider a sequence of terms is formed by following the two rules below:

   (i) If a term is even, divide by 2 to get the next term.

   (ii) If a term is odd, multiply by 3, then add 1, to get the next term.

   Based on the definition, answer the following questions:

   (a) Choose a positive integer of your liking as the first term and compute the next few terms of the sequence. Does your sequence exhibit any interesting behavior?

   (b) When we start with 12, the first few terms of the sequence are 12, 6, 3, 10, 5, 16, $\ldots$. What is the sum of the sequence's first 1000 terms?

3. (Continuation) Does the choice of seed value (first term) affect the difficulty of finding the sum in the previous question? Before you answer, you might want to consider the seed value 27.

   Let the first term be T_0. Write a recursive relation of T_{i+1} in terms of T_i for every positive integer i.

   You may wonder what is the Collatz conjecture? It asks:

   > Given any starting value $n > 0$, does the sequence T_i eventually contain 1?

   So we don't know if for any positive integral seed value n, we eventually get to the number 1.

4. Given that $x_1 = 211$, $x_2 = 375$, $x_3 = 420$, $x_4 = 523$, and $x_n = x_{n-1} - x_{n-2} + x_{n-3} - x_{n-4}$, when $n \geq 5$, find the value of $x_{531} + x_{753} + x_{975}$.

5. A dresser has eight drawers stacked vertically. To be able to reach the contents in an open drawer, no drawer that is adjacent to the open drawer may be open at the same time. In how many ways can one or more drawers be open so that the contents in each of the open drawers can be reached?