Lectures on Challenging Mathematics

Integrated Mathematics 5

Geometry

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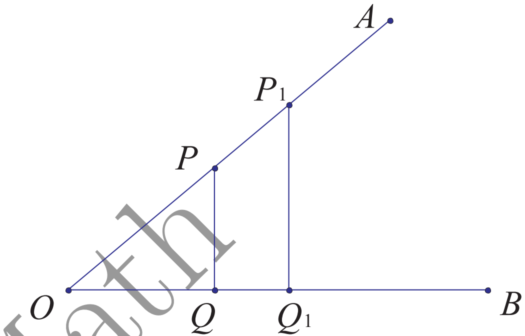
1.6 Right triangle trigonometry (part 3)

1. Let rays OA and OB form angle θ. Choose point P on ray OA. Let Q be the *foot* (that is the bottom) of the perpendicular line segment from P to the ray OB.

   We define the *sine function* (sin) as follows,

   $$\sin\theta = \frac{PQ}{OP}.$$

   We need to show that the sine function is *well defined*; that is, its value only depends on the size of θ, but not the choice of P. To do so, let P_1 denote another point lying on ray OA, and let Q_1 be the foot of perpendicular from P_1 to ray OB. Explain why right triangles OPQ and OP_1Q_1 are similar to each other, and hence pairs of corresponding ratios, such as $\frac{PQ}{OP}$ and $\frac{P_1Q_1}{OP_1}$, are all equal. Therefore, the sine function is indeed well defined.

   Evaluate $\sin 30°$, $\sin 45°$, and $\sin 60°$. Check your answers with a calculating device.

2. A ladder leans against a side of a building, making a 63-degree angle with the ground, and reaching over a fence that is 6 feet from the building. The ladder barely touches the top of the fence, which is 8 feet tall. Find the length of the ladder.

3. A line of positive slope is drawn so that it makes a 60-degree angle where it intersects the x-axis. Peter and Curtis were asked to find the slope of this line. Peter said, "If the angle were 45-degree, the slope would be 1. I think the slope for the 60-degree case is 4/3, which is equal to 60/45." Curtis commented, "I think there are some flaws in your reasoning. If I replace 60-degree by some other angle in your method, I get some obviously wrong answer." What angle does Curtis had in mind? How to solve this problem correctly?

4. One day at the beach, Kelly flies a kite, whose string makes a 37-degree elevation angle with the ground. The length of the string is 168 feet. How high above the ground is the kite, to the nearest foot? What (unrealistic) assumptions did you make in answering this question?

5. Find the angles in triangle ABC where

   (a) $A = (0, 1)$, $B = (0, 9)$, $C = (5, 4)$; (b) $A = (1, 2)$, $B = (7, 5)$, $C = (5, -9)$.

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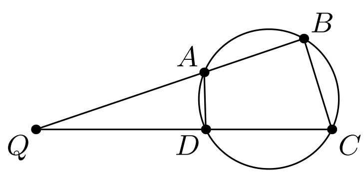
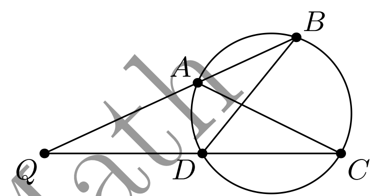
1.19 Arcs and angles (part 5)

1. Two of the tangents to a circle meet at Q, which is 25 cm from the center. The circle has a 7 cm radius. To the nearest tenth of a degree, find the angle formed at Q by the tangents.

2. *Crossed Chords Theorem Revisited.* Suppose that A, B, C, D lie (in that order) on a circle, and that secant lines AB and CD meet in a point Q outside the circle.





   (a) Spot a pair of similar triangles in each of the diagrams shown.

   (b) Show that $QA \cdot QB = QC \cdot QD$.

3. The segments GA and GB are tangent to a circle at A and B, and AGB is a 60-degree angle. Given that $GA = 12$ cm, find the distance from G to the nearest point on the circle.

4. Suppose that MP is a diameter of a circle centered at O, and Q is any other point on the circle. Draw the line through O that is parallel to MQ, and let R be the point where it meets minor arc PQ. Prove that R is the midpoint of minor arc PQ.

5. In quadrilateral $FAIR$, $FA = 15$, $AI = 24$, $IR = 7$, $RF = 20$, and $AR = 25$. Diagonals AR and FI meet in P. Find the distance from P to the midpoint of segment AR.

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