Lectures on Challenging Mathematics

Integrated Mathematics 5

Algebra

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Zuming Feng
Phillips Exeter Academy and IDEA Math
zfeng@exeter.edu

Contents

1.3 Arithmetic and geometric progressions (part 2)

1. Call a three-digit number *geometric* if it has three distinct digits which, when read from left to right, form a geometric sequence. Find the difference between the largest and smallest geometric numbers.

2. When asked to evaluate the sum of an arithmetic series, Blair claimed "That's easy – just take average of the first and last terms, then multiply by the number of terms." Explain Blair's strategy.

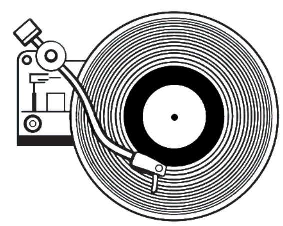
   Consider now the typical arithmetic series, which looks like

   $$\text{first} + (\text{first} + \text{difference}) + (\text{first} + 2 \cdot \text{difference}) + \cdots + \text{last}.$$

   Find a compact formula for this series in terms of *first* (f), *last* (ℓ), and *difference* (d).

3. A triangle with a 32-inch side, a 40-inch side, and a 50-inch side is a curiosity, for its sides form a geometric sequence. Find the constant multiplier for this sequence. Find other such triangles. Are there any restrictions on the multipliers that can be used?

4. A typical long-playing phonograph record (once known as an LP) plays for about 24 minutes at 33 1/3 revolutions per minute while a needle traces the long groove that spirals slowly in towards the center. The needle starts 5.7 inches from the center and finishes 2.5 inches from the center. Estimate the length of the groove. Why it is an *estimation* rather than an accurate answer? In other words, what assumptions did you make in answering this question?



5. For the first 31 days of your new job, your boss offers you two salary options. The first option pays you \$1000 on the first day, \$2000 on the second day, \$3000 on the third day, and so on — in other words, \$1000n on the n^{th} day. The second option pays you one penny on the first day, two pennies on the second day, four pennies on the third day — the amount doubling from one day to the next. Which option do you prefer, and why?

   Now we assume that you have chosen the second payment option. On the thirty-first day your boss pays you the wages for that day — in pennies. You wonder whether all these coins are going to fit into your dormitory room, which measures 12 feet by 15 feet by 8 feet. Verify that a penny is 0.75 inch in diameter, and that seventeen of them make a stack that is one inch tall. Use this information to decide whether the pennies will all fit.

   What is the exact number of pennies you will be paid after 31 days at your job?

1.16 Operations rules with logarithms (part 5)

1. Without calculator, solve for real x:

   (a) $\log_4 x = -1.5$
   (b) $\log_x 8 = 6$
   (c) $27 = 8(x-2)^3$
   (d) $3\log_{27}(x-2)^4 = 2$

2. Rewrite the equation $19\log x - 9 + 9\log y = 199\log(8z)$ so that it makes no reference to logarithms.

3. When $10^{3.43429448}$ is evaluated, how many digits are found to the left of the decimal point? You can answer this question without using your calculator. Use your calculator to find its leftmost three digits.

   What are the leftmost three digits when each of the following numbers is evaluated?

   (a) $10^{7.43429448}$
   (b) $10^{-0.43429448}$
   (c) $10^{0.56570552}$
   (d) $10^{-7.4342944}$

4. Earthquakes can be classified by the amount of energy they release. Because of the large numbers involved, this is usually done logarithmically. The Richter scale is defined by the equation $R = 0.67\log(E) - 1.17$, where R is the rating and E is the energy carried by the seismic wave, measured in kilowatt-hours. (A kilowatt-hour is the energy consumed by ten 100-watt light bulbs in an hour.)

   (a) The 1989 earthquake in San Francisco was rated at 7.1. What amount of energy did this earthquake release? It could have sustained how many 100-watt light bulbs for a year?
   (b) An earthquake rated at 8.1 releases more energy than an earthquake rated at 7.1. How many times more?
   (c) Rewrite the defining equation so that E is expressed as a function of R.
   (d) Adding 1 to any rating corresponds to multiplying the energy by what constant?
   (e) Is it possible for a seismic wave to have a *negative* rating? What would that signify?

5. Solve each of the following equations for x.

   (a) $(\log x)^2 + \log(x^2) = \frac{5}{4}$
   (b) $2\log(2x)^3 = 3\log(x-15)^2$