

Lectures on Challenging Mathematics

Integrated Mathematics 3

Geometry (Part 1)

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Zuming Feng

Phillips Exeter Academy and IDEA Math

zfeng@exeter.edu

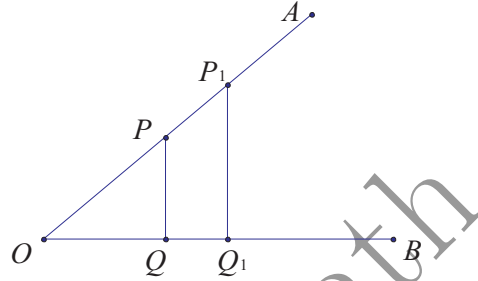
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1.6 Right triangle trigonometry (part 3)

- Let rays OA and OB form angle θ . Choose point P on ray OA . Let Q be the *foot* (that is the bottom) of the perpendicular line segment from P to the ray OB .



We define the *sine function* (\sin) as follows,

$$\sin \theta = \frac{PQ}{OP},$$

We need to show that the sine function is *well defined*; that is, its value only depends on the size of θ , but not the choice of P . To do so, let P_1 denote another point lying on ray OA , and let Q_1 be the foot of perpendicular from P_1 to ray OB . (By the way, “ P sub 1” is how P_1 is usually read.) Explain why right triangles OPQ and OP_1Q_1 are similar to each other, and hence pairs of corresponding ratios, such as $\frac{PQ}{OP}$ and $\frac{P_1Q_1}{OP_1}$, are all equal. Therefore, the sine function is indeed well defined.

Evaluate $\sin 30^\circ$, $\sin 45^\circ$, and $\sin 60^\circ$. Check your answers with a calculating device.

- A ladder leans against a side of a building, making a 63-degree angle with the ground, and reaching over a fence that is 6 feet from the building. The ladder barely touches the top of the fence, which is 8 feet tall. Find the length of the ladder.
- One day at the beach, Kelly flies a kite, whose string makes a 37-degree elevation angle with the ground. The length of the string is 168 feet. How high above the ground is the kite, to the nearest foot? What (unrealistic) assumptions did you make in answering this question?
- To draw a right triangle that has a 1-degree angle and measure its sides accurately is difficult. To get the sine ratio for a 1-degree angle, however, there is an easy way – just use a calculator. Is the ratio a small or large number? How large can a sine ratio be?

Use your calculator to evaluate $\tan 85^\circ$, $\tan 89^\circ$, $\tan 89.99^\circ$, $\sin 89^\circ$, $\sin 0.5^\circ$? Do these values make sense to you? What can you say about $\tan 90^\circ$, $\sin 90^\circ$, and $\sin 0^\circ$?

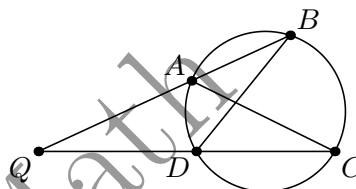
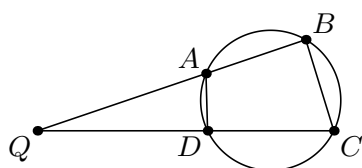
- Find the angles in triangle ABC where

(a) $A = (0, 1)$, $B = (0, 9)$, $C = (5, 4)$;

(b) $A = (1, 2)$, $B = (7, 5)$, $C = (5, -9)$.

1.19 Arcs and angles (part 5)

- Two of the tangents to a circle meet at Q , which is 25 cm from the center. The circle has a 7 cm radius. To the nearest tenth of a degree, find the angle formed at Q by the tangents.
- Crossed Chords Theorem Revisited.* Suppose that A, B, C, D lie (in that order) on a circle, and that secant lines AB and CD meet in a point Q outside the circle.



- Determine if any of $AB \parallel CD$ and $AD \parallel BC$ is always true?
 - Spot a pair of similar triangles in each of the diagrams shown.
 - Show that $QA \cdot QB = QC \cdot QD$.
- The segments GA and GB are tangent to a circle at A and B , and AGB is a 60-degree angle. Given that $GA = 12$ cm, find the distance from G to the nearest point on the circle.
 - Suppose that MP is a diameter of a circle centered at O , and Q is any other point on the circle. Draw the line through O that is parallel to MQ , and let R be the point where it meets minor arc PQ . Prove that R is the midpoint of minor arc PQ .
 - In quadrilateral $FAIR$, $FA = 15$, $AI = 24$, $IA = 7$, $RF = 20$, and $AR = 25$. Diagonals AR and FI meet in P . Find the distance from P to the midpoint of segment AR .