

# Lectures on Challenging Mathematics

## Math Challenges 4

### Algebra

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## 1.8 Arithmetic and algebra techniques (part 4)

1. A rectangle has area  $\frac{T}{4} + \frac{1}{8}$  and a diagonal of length  $\frac{T}{2}$ . Express the perimeter of the rectangle in terms of  $T$ .
2. Consider the sequence

$$s_1 = 1 + 2, \quad s_2 = 1 + 2 + 2^2, \quad s_3 = 1 + 2 + 2^2 + 2^3, \dots$$

- (a) Evaluate  $s_1, s_2, s_3, s_4, s_5$ . Do you observe any pattern? Check your pattern with the actual values of  $s_6$  and  $s_7$ .
  - (b) Explain the reason behind your observation. You might want to consider either  $1 + s_n$  or  $s_n + 1$  – even though these two expressions are equal to each other, one is more helpful than the other.
  - (c) Factor  $s_{15}$  and  $s_{16}$ .
3. Odell and Kershaw run for 30 minutes on a circular track. Odell runs clockwise at 250 m/min and uses the inner lane with a radius of 50 meters. Kershaw runs counterclockwise at 300 m/min and uses the outer lane with a radius of 60 meters, starting on the same radial line as Odell. How many times after the start do they pass each other?
  4. Find all integers  $n$  such that  $p(n)$  is a perfect square where
    - (a)  $p(n) = n^2 + 24$ ;
    - (b)  $p(n) = n^2 + 6n - 212$ ;
    - (c)  $p(n) = n^2 - 20n - 33$ ;
    - (d)  $p(n) = n^2 - 19n + 91$ .
  5. If  $x, y$ , and  $z$  are positive real numbers satisfying

$$x + \frac{1}{y} = 4, \quad y + \frac{1}{z} = 1, \quad \text{and} \quad z + \frac{1}{x} = \frac{7}{3},$$

then what is  $xyz$ ?