Lectures on Challenging Mathematics

Math Challenges 4

Algebra

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1.8 Arithmetic and algebra techniques (part 4)

1. A rectangle has area $\frac{T}{4}+\frac{1}{8}$ and a diagonal of length $\frac{T}{2}$. Express the perimeter of the rectangle in terms of T.

2. Consider the sequence

$$s_1 = 1+2, \quad s_2 = 1+2+2^2, \quad s_3 = 1+2+2^2+2^3, \ldots$$

   (a) Evaluate s_1, s_2, s_3, s_4, s_5. Do you observe any pattern? Check your pattern with the actual values of s_6 and s_7.

   (b) Explain the reason behind your observation. You might want to consider either $1+s_n$ or s_n+1 – even though these two expressions are equal to each other, one is more helpful than the other.

   (c) Factor s_{15} and s_{16}.

3. Odell and Kershaw run for 30 minutes on a circular track. Odell runs clockwise at 250 m/min and uses the inner lane with a radius of 50 meters. Kershaw runs counterclockwise at 300 m/min and uses the outer lane with a radius of 60 meters, starting on the same radial line as Odell. How many times after the start do they pass each other?

4. Find all integers n such that $p(n)$ is a perfect square where

   (a) $p(n) = n^2 + 24$;
   (b) $p(n) = n^2 + 6n - 212$;
   (c) $p(n) = n^2 - 20n - 33$;
   (d) $p(n) = n^2 - 19n + 91$.

5. If x, y, and z are positive real numbers satisfying

$$x + \frac{1}{y} = 4, \quad y + \frac{1}{z} = 1, \quad \text{and} \quad z + \frac{1}{x} = \frac{7}{3},$$

then what is xyz?