Lectures on Challenging Mathematics

Olympiad Math 2

Number Theory

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1.16 Modular arithmetic (part 6)

1. When $n = 1$ clearly $2^n - 2$ is divisible by 7, $4^n - 4$ is divisible by 9, $6^n - 6$ is divisible by 11, and $8^n - 8$ is divisible by 13. Find the next smallest positive integer n for which this occurs.

2. Find, with the proof, the smallest integer n such that for any prime p

$$\Big(1! \cdot 2! \cdots (p-1)!\Big)^n \equiv 1 \pmod{p}.$$

3. Let $m > 1$ be a positive integer. Find all m such that numbers in the set $Q = \{1^2, 2^2, \ldots, m^2\}$ give $\lfloor \frac{m}{2} \rfloor + 1$ different remainders modulo m.

4. Show that if we consider the numbers in the set $\{1!, 2!, \ldots, (p-1)!\}$ modulo p there are at least $\sqrt{p}$ different residues modulo p.

5. Prove that there exists a positive integer $n < 10^6$ such that 5^n has six consecutive zeros in its decimal representation.