Lectures on Challenging Mathematics

Olympiad Math 2

Combinatorics

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1. Determine the number of permutations π of $(1, 2, \ldots, 10)$ satisfying the following property:

   (a) For every $1 \le i \le 10$, $\pi^{(30)}(i) = i$;

   (b) For every $1 \le k \le 29$, there is a $1 \le i \le 10$ such that $\pi^{(k)}(i) \ne i$.

   (For $n \ge 1$, $\pi^{(n+1)}(i) = \pi(\pi^{(n)}(i))$.)

2. Prove that one can construct two triangles by using all the edges of an arbitrary tetrahedron.

3. At a conference, each participant is acquainted with at least one other participant, and for each pair of participants there is a participant not acquainted to both. Prove that one can divide all the participants into three nonempty groups so that each participant would be acquainted with at least one person in his group.

4. Fifteen rooks are placed on a regular chessboard satisfying the following conditions:

   > each unit field is occupied by at most one rook and there is at least one rook in each row and each column.

   Prove that one can remove one of the rooks so that the remaining rooks still satisfy the above conditions.

   Do you see the relation between this problem and the following problem:

   > Find the smallest positive integer n such that if n unit squares of a 1000×1000 unit-square board are colored, then there will exist three colored unit squares whose centers form a right triangle with legs parallel to the edges of the board.

5. Determine if there is a positive integer n such that the four most significant digits of n are 2015; that is, $n! = 2015\ldots$.