Lectures on Challenging Mathematics

Olympiad Math 1

Combinatorics

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1.17 Challenges in enumerative counting (part 8)

1. A *look-and-say sequence* is defined as follows: starting from an initial term a_1, each subsequent term a_k is found by reading the digits of a_{k-1} from left to right and specifying the number of times each digit appears consecutively. For example, 4 would be succeeded by 14 ("One four."), and 31337 would be followed by 13112317 ("One three, one one, two three, one seven.")

   If a_1 is a random two-digit positive integer, find the probability that a_4 is at least six digits long.

2. A classroom has 2 rows and 11 columns of desks, and there is one student at each of the 22 desks. The teacher wishes to reassign the seats in such a way that each student finds a neighboring student (that is, a student sitting one seat to the left or right, or one seat forward or backward) and swaps seat with this neighbor. (Each student swaps his seat with exactly one of his neighbors once.) In how many ways can this be done?

3. A vertical polygon path will be formed by picking one point from each row of a 4×4 grid of points (the left-hand side figure shown below), and then connecting these points sequentially form top to bottom. Such a path is called *balanced* if the area of the region within the grid to the left of the path is equal to that of the region within the grid to the right of the path. For how many 4-point selections will be the resulting polygon path being balanced? (One balanced path is shown in the right-hand side figure below.)

4. Let n and k be positive integers with $k + 3 \leq n$. Prove that $\binom{n}{k}$, $\binom{n}{k+1}$, $\binom{n}{k+2}$, $\binom{n}{k+3}$ cannot form an arithmetic progression.

5. On the Exeter Space Station, where there is no effective gravity, Chad has a geometric model consisting of 125 wood cubes measuring 1 centimeter on each edge arranged in a 5 by 5 by 5 cube. An aspiring carpenter, he practices his trade by drawing the projection of the model from three views: front, top, and side. Then, he removes some of the original 125 cubes and redraws the three projections of the model. He observes that his three drawings after removing some cubes are identical to the initial three. What is the maximum number of cubes that he could have removed? (Keep in mind that the cubes could be suspended without support.