Lectures on Challenging Mathematics

Olympiad Math 1

Algebra

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Contents

1.15 Basic properties of polynomials

1. [Lagrange's Interpolation Formula] There is a unique second degree polynomial $p(x)$ passing through points $(1,5),(3,8),(6,-7)$. Explain why

$$p(x)=\frac{5(x-3)(x-6)}{(1-3)(1-6)}+\frac{8(x-1)(x-6)}{(3-1)(3-6)}-\frac{7(x-1)(x-3)}{(6-1)(6-3)}.$$

Find a third degree polynomial that passes through points $(1,0)$, $(2,1)$, $(4,14)$, and $(6,55)$.

2. (Continuation) In general, let $x_0,x_1,\ldots,x_n$ be distinct real numbers, and let $y_0,y_1,\ldots,y_n$ be arbitrary real numbers. Then there exists a unique polynomial $P(x)$ of degree at most n such that $P(x_i)=y_i$, $i=0,1,\ldots,n$. Show that this polynomial is

$$P(x)=\sum_{i=0}^{n} y_i\frac{(x-x_0)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}.$$

3. Let $P(x)$ be a polynomial with leading coefficient 1 and integer coefficients. If u and v are positive integers, where v is not a perfect square, and $u+\sqrt{v}$ is a root of $P(x)$, show that $u-\sqrt{v}$ is also a root of $P(x)$.

4. Let $f(x)=x^4-49x^2-14x-1$ and let $g(x)=ax+b$. Find positive integers a and b for which $f(g(x))$ is divisible by $x^2+9x+19$.

5. The polynomial P is a quadratic with integer coefficients. For every positive integer n, the integers $P(n)$ and $P(P(n))$ are relatively prime to n. If $P(3)=89$, determine with justification the value of $P(10)$?