

Lectures on Challenging Mathematics

Math Challenges 7

Number Sense

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Internal Use

1.7 A beginners' tour of the Diophantine equation - a short review

1. Sarah intended to multiply a two-digit number and a three-digit number, but she left out the multiplication sign and simply placed the two-digit number to the left of the three-digit number, thereby forming a five-digit number. This number is exactly nine times the product Sarah should have obtained. What is the sum of the two-digit number and the three-digit number?
2. How many ordered four-tuples of integers (a, b, c, d) with $0 < a < b < c < d < 500$ satisfy $a + d = b + c$ and $bc - ad = 93$?

3. What is the largest positive integer n for which there is a unique integer k such that

$$\frac{8}{15} < \frac{n}{n+k} < \frac{7}{13}?$$

4. In the following 3×3 array of distinct positive integers, the products of entries in each row, each column, and each diagonal are all equal to n . Find the least possible value of n .

$$\begin{array}{ccc} a & b & c \\ d & e & f \\ g & h & i \end{array}$$

5. For each integer $a_0 > 1$, define the sequence (a_0, a_1, a_2, \dots) by

$$a_{n+1} = \begin{cases} \sqrt{a_n}, & \text{if } \sqrt{a_n} \text{ is an integer;} \\ a_n + 3, & \text{otherwise} \end{cases}$$

Determine all possible values of a_0 for which the sequence eventually is strictly increasing.