Lectures on Challenging Mathematics

Math Challenges 7

Geometry

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1.7 Revisiting arcs and angles

1. Let $P_1, P_2, \ldots, P_n$ are evenly distributed around a circle. Determine the minimum value of $n$, given that there are three points $P_i, P_j, P_k$ such that in triangle $P_iP_jP_k$

(a) $\angle P_i = \frac{180^\circ}{7}$, $\angle P_j = \frac{360^\circ}{7}$, $\angle P_k = \frac{720^\circ}{7}$

(b) $\angle P_i = 40^\circ$, $\angle P_j = 60^\circ$, $\angle P_k = 80^\circ$

2. In triangle $ABC$, we have $AB = 7$, $AC = 8$, and $BC = 9$. Point $D$ lies on the circumcircle of triangle $ABC$ so that ray $AD$ bisects $\angle BAC$. What is the value of $AD/CD$?

3. Let $M$ and $A$ be two given points on circle $\omega$ with minor arc $\overline{MA} = 80^\circ$. Let $T$ and $H$ be two moving points on the major $\overline{MA}$ with minor arc $\overline{TH} = 100^\circ$. Diagonals of the quadrilateral $MATH$ meet at $P$. As $T$ and $H$ moving along the arc, what is the locus of $P$?

4. Distinct points $A$ and $B$ are on a semicircle with diameter $MN$ and center $C$. Point $P$ lies on segment $CN$ and $\angle CAP = \angle CBP = \alpha$ and $\angle ACM = \beta$. Suppose that $A$ lies on $\overline{MB}$, express $\angle BPN$ in terms of $\alpha$ and $\beta$.

5. (Continuation) Distinct points $A$ and $B$ are on a semicircle with diameter $MN$ and center $C$. Point $P$ lies on segment $CN$ and $\angle CAP = \angle CBP = \alpha$ and $\angle ACM = \beta$. Suppose that $B$ lies on $\overline{MA}$, express $\angle BPN$ in terms of $\alpha$ and $\beta$. 