Lectures on Challenging Mathematics

Math Challenges 7

Geometry

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1.7 Revisiting arcs and angles

1. Let $P_1, P_2, \ldots, P_n$ are evenly distributed around a circle. Determine the minimum value of n, given that there are three points P_i, P_j, P_k such that in triangle $P_iP_jP_k$

   (a) $\angle P_i = \dfrac{180^\circ}{7}$, $\angle P_j = \dfrac{360^\circ}{7}$, $\angle P_k = \dfrac{720^\circ}{7}$ (b) $\angle P_i = 40^\circ$, $\angle P_j = 60^\circ$, $\angle P_k = 80^\circ$

2. In triangle ABC, we have $AB = 7$, $AC = 8$, and $BC = 9$. Point D lies on the circumcircle of triangle ABC so that ray AD bisects $\angle BAC$. What is the value of AD/CD?

3. Let M and A be two given points on circle ω with minor arc $\widehat{MA} = 80^\circ$. Let T and H be two moving points on the major $\widehat{MA}$ with minor arc $\widehat{TH} = 100^\circ$. Diagonals of the quadrilateral $MATH$ meet at P. As T and H moving along the arc, what is the locus of P?

4. Distinct points A and B are on a semicircle with diameter MN and center C. Point P lies on segment CN and $\angle CAP = \angle CBP = \alpha$ and $\angle ACM = \beta$. Suppose that A lies on $\widehat{MB}$, express $\angle BPN$ in terms of α and β.

5. (Continuation) Distinct points A and B are on a semicircle with diameter MN and center C. Point P lies on segment CN and $\angle CAP = \angle CBP = \alpha$ and $\angle ACM = \beta$. Suppose that B lies on $\widehat{MA}$, express $\angle BPN$ in terms of α and β.