Lectures on Challenging Mathematics

Math Challenges 7

Counting

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1.5 Bijections

1. Determine the number of ways to choose five numbers from the first eighteen positive integers such that any two chosen numbers differ by at least 2.

2. Determine the number of subsets S of $\{1, 2, 3, \ldots, 10\}$ with the following property: there exist integers $a < b < c$ with $a \in S$, $b \notin S$, $c \in S$.

3. Every card in a deck has a picture in one shape–a circle, a square, or a triangle, which is painted in one of three colors–red, blue, or green. Furthermore, each color is applied in one of three shades–light, medium, or dark. The deck has 27 cards, with every shape–color–shade combination represented. A set of three cards from the deck is called *complementary* if all of the following statements are true:

   (a) Either each of the three cards has a different shape or all three cards have the same shape.

   (b) Either each of the three cards has a different color or all three cards have the same color.

   (c) Either each of the three cards has a different shade or all three cards have the same shade.

   How many different complementary three-card sets are there?

4. [By Weichao Wu] Let n be an integer with $n \geq 2$, and define the sequence $S = (1, 2, \ldots, n)$. A subsequence of S is called arithmetic if it has at least two terms and it is an arithmetic progression. An arithmetic subsequence is called maximal if this progression cannot be lengthened (at either ends) by the inclusion of another element of S. Determine the number of maximal arithmetic subsequences.

5. Draw $2m$ points $P_1, P_2, \ldots, P_{2m}$ equally spaced around the circumference of a circle.

   (a) How many (unordered) triples $\{P_i, P_j, P_k\}$ are there so that triangle $P_iP_jP_k$ is acute? obtuse?

   (b) What is the ratio between the number of acute triangles and the number of obtuse triangles? Hmm ...