Lectures on Challenging Mathematics

Math Challenges 7

Algebra

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Contents

1 Algebra **3**
1.1 Recursive sequences (part 1) . . . 3
1.2 Absolute value functions . . . 4
1.3 Systems of logarithm equations (part 1) . . . 5
1.4 Quadratic equations . . . 6
1.5 Algebra practice set 1 . . . 7
1.6 Solving equations (part 1) . . . 8
1.7 Recursive sequences (part 2) . . . 9
1.8 Solving equations (part 2) . . . 10
1.9 Systems of logarithm equations (part 2) . . . 11
1.10 Algebra practice set 2 . . . 12

1.7 Recursive sequences (part 2)

1. Consider sequences of positive real numbers of the form $x, 2000, y, \ldots$, in which every term after the first is 1 less than the product of its two immediate neighbors. For how many different values of x does the term 2001 appear somewhere in the sequence?

2. Let $P_0(x) = x^3 + 313x^2 - 77x - 8$. For integers $n \geq 1$, define $P_n(x) = P_{n-1}(x - n)$. What is the coefficient of x in $P_{20}(x)$?

3. The sequence $\{a_n\}$ is defined by

$$a_0 = 1,\ a_1 = 1,\ \text{and } a_n = a_{n-1} + \frac{a_{n-1}^2}{a_{n-2}} \text{ for } n \geq 2.$$

The sequence $\{b_n\}$ is defined by

$$b_0 = 1,\ b_1 = 3,\ \text{and } b_n = b_{n-1} + \frac{b_{n-1}^2}{b_{n-2}} \text{ for } n \geq 2.$$

Find $\dfrac{b_{32}}{a_{32}}$.

4. Let m be a positive integer, and let $a_0, a_1, \ldots, a_m$ be a sequence of real numbers such that $a_0 = 37$, $a_1 = 72$, $a_m = 0$ and

$$a_{k+1} = a_{k-1} - \frac{3}{a_k}$$

$k = 1, 2, \ldots, m-1$. Find m.

5. Let A, B, C, and D be the vertices of a regular tetrahedron, each of whose edges measures 1 meter. A bug, starting from vertex A, observes the following rule: At each vertex it chooses one of the three edges meeting at that vertex, each edge being equally likely, and crawls along that edge to the vertex at its opposite end. Find the probability that the bug is at vertex A when it has crawled exactly 7 meters.