Lectures on Challenging Mathematics

Integrated Math 7A

Analytic Geometry and Vector Operations

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1.13 Vector motion (part 5)

1. A *direction vector* for a line is any vector that joins two points on that line. A *normal vector* of a line is any vector that is perpendicular to the line.

   (a) Find a direction vector and a normal vector for $2x + 5y = 8$. It is not certain that you and your classmates will get exactly the same answer. How should your answers be related, however?

   (b) Show that $[b, -a]$ is a direction vector for the line $ax + by = c$.

   (c) Show that $[a, b]$ is a normal vector for the line $ax + by = c$.

2. (Continuation) In 2-D (coordinate) geometry, one might think that the concepts of direction vector and normal vector are equally important and maybe somewhat redundant. But in 3-D (coordinate) geometry, these two concepts could be vastly different. In particular, which of the following definitions are meaningful?

   (a) A *direction vector* for a *plane* is any vector that joins two points on that plane.

   (b) A *normal vector* of a *line* is any vector that is perpendicular to the line.

   (c) A *normal vector* of a *plane* is any vector that is perpendicular to the plane.

   Use your imagination to define a *direction vector* for a *curve* at a given point on the curve.

3. If M is the midpoint of segment AB, how are vectors $\overrightarrow{AM}$, $\overrightarrow{AB}$, $\overrightarrow{MB}$, and $\overrightarrow{BM}$ related?

4. Let $A = (5, -3, 6)$, $B = (0, 0, 0)$, and $C = (3, 7, 1)$. Show that

   (a) $\angle ABC$ is a right angle.

   (b) the vectors $[5, -3, 6]$ and $[3, 7, 1]$ are perpendicular.

5. Two of the midpoints of a triangle are $(3, -1)$ and $(4, 3)$. One of the vertices of the triangle is $(7, -3)$.

   (a) Find the other vertices of one such triangles.

   (b) How many such triangles are there? Find the vertices of each of these triangles.

1.17 Analytic transformations (part 2)

1. Apply the transformation $\mathcal{T}(x, y) = (2x + 3y, -x + y)$ to the unit square, whose vertices are $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$. Even though $\mathcal{T}$ is not a reflection, it is customary to call the resulting figure the *image* of the square. What kind of figure is it? Is this transfromation an isometry?

2. Decide whether each of the following transformation is an isometry, and give your reasons.

   (a) $\mathcal{T}_1(x, y) = \left(2x, \dfrac{1}{2}y\right)$ (b) $\mathcal{T}_2(x, y) = (-x, y + 2)$

3. Given the transformation $\mathcal{F}(x, y) = (-0.6x - 0.8y, 0.8x - 0.6y)$, Shane calculated the image of the isosceles right triangle formed by $S = (0, 0)$, $H = (0, -5)$, and $A = (5, 0)$, and declared that $\mathcal{F}$ is a reflection. Morgan instead calculated the image of the *scalene* triangle formed by $M = (7, 4)$, $O = (0, 0)$, and $R = (7, 1)$, and concluded that $\mathcal{F}$ is a rotation. Who was correct? Explain your choice, and account for the disagreement.

4. Determine if $\mathcal{F}$ is an isometry.

5. Explain why an isometry always transforms a right triangle onto a right triangle. In particular, does isometry preserve angle measures? If so, why this information is not part of the definition?

1.30 Lattice points (part 5)

1. In triangle ABC, $A = (0, 0)$, $B = (4, 3)$, $C = (5, 12)$. Point D lies on side BC such that $\angle BAD = \angle CAD$. Find the coordinates of D.

2. Let $P = (-15, 0)$, $Q = (5, 0)$, $R = (8, 21)$, and $S = (0, 15)$. Draw quadrilateral $PQRS$ and measure its sides and angles. Is there anything remarkable about this figure? Make a conjecture from your observation and use vector operations or lattice points wise to confirm your conjecture.

3. Describe a transformation that carries the triangle with vertices $K = (0, 0)$, $L = (10, 5)$, and $M = (6, 18)$ onto the triangle with vertices $R = (3, 4)$, $P = (-2, 14)$, and $Q = (9, 22)$. Where does your transformation send $(5, 3)$?

4. Suppose that a quadrilateral is measured and found to have one of the following set of properties. Is this evidence enough to conclude that the quadrilateral is a parallelogram? Explain.

   (a) two pairs of equal nonadjacent sides

   (b) two pairs of equal nonadjacent angles

   (c) a pair of equal nonadjacent sides and a pair of equal nonadjacent angles

5. In a coordinate plane, determine if it is possible for an equilateral triangle to have all its vertices to be lattice points.