Lectures on Challenging Mathematics

Integrated Mathematics 4

Analytic geometry and vector operations

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Contents

1.13 Vector motion (part 5)

1. A *direction vector* for a line is any vector that joins two points on that line. A *normal vector* of a line is any vector that is perpendicular to the line.

   (a) Find a direction vector and a normal vector for $2x + 5y = 8$. It is not certain that you and your classmates will get exactly the same answer. How should your answers be related, however?

   (b) Show that $[b, -a]$ is a direction vector for the line $ax + by = c$.

   (c) Show that $[a, b]$ is a normal vector for the line $ax + by = c$.

2. (Continuation) In 2-D (coordinate) geometry, one might think that the concepts of direction vector and normal vector are equally important and maybe somewhat redundant. But in 3-D (coordinate) geometry, these two concepts could be vastly different. In particular, which of the following definitions are meaningful?

   (a) A *direction vector* for a *plane* is any vector that joins two points on that plane.

   (b) A *normal vector* of a *line* is any vector that is perpendicular to the line.

   (c) A *normal vector* of a *plane* is any vector that is perpendicular to the plane.

   Use your imagination to define a *direction vector* for a *curve* at a given point on the curve.

3. If M is the midpoint of segment AB, how are vectors $\overrightarrow{AM}$, $\overrightarrow{AB}$, $\overrightarrow{MB}$, and $\overrightarrow{BM}$ related?

4. Translate the line $5x + 7y = 35$ by vector $[3, 10]$. Find an equation for the new line.

   Find a vector that translates the line $2x - 3y = 18$ onto the line $2x - 3y = 24$.

   Which of the above questions has more than one correct answer. Can you describe all those answers?

5. Two of the midpoints of a triangle are $(3, -1)$ and $(4, 3)$. One of the vertices of the triangle is $(7, -3)$.

   (a) Find the other vertices of one such triangles.

   (b) How many such triangles are there? Find the vertices of each of these triangles.

1.21 Lattice points (part 5)

1. Consider two convex equilateral octagons satisfying the following properties:

   - they have equal side length;
   - they are not congruent to each other;
   - all their vertices are lattice points.

   Determine the minimum side length of the two octagons.

2. Describe a transformation that carries the triangle with vertices $K = (0, 0)$, $L = (10, 5)$, and $M = (6, 18)$ onto the triangle with vertices $R = (3, 4)$, $P = (-2, 14)$, and $Q = (9, 22)$. Where does your transformation send $(5, 3)$?

3. Suppose that a quadrilateral is measured and found to have one of the following set of properties. Is this evidence enough to conclude that the quadrilateral is a parallelogram? Explain.

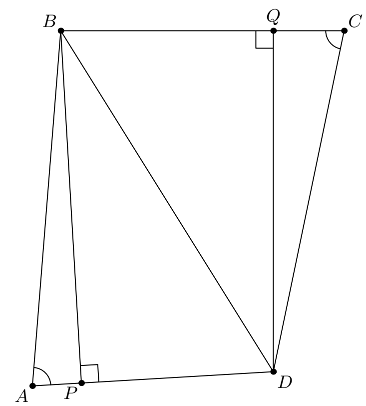
   (a) two pairs of equal nonadjacent sides

   (b) two pairs of equal nonadjacent angles

   (c) a pair of equal nonadjacent sides and a pair of equal nonadjacent angles

4. (Continuation) Max thinks that for part (c) the other two nonadjacent sides must be always equal. His argument goes as follows:

   Let $ABCD$ be a quadrilateral such that $\angle A = \angle C$ and $AB = CD$. Denote points P and Q on AD and BC, respectively, such that $BP \perp AD$ and $DQ \perp BC$. Triangles ABP and CDQ are congruent, so $AP = CQ$ and $BP = DQ$. Right triangles BPD and DQB are also congruent, yielding $DP = BQ$. Thus $AD = AP + DP = CQ + BQ = BC$.

   Do you agree with Max's argument?

5. In a coordinate plane, determine if it is possible for an equilateral triangle to have all its vertices to be lattice points.

1.24 3-D rectangular coordinates and linear equations (part 2)

1. The edges of rectangular box $ABCDEFGH$ are parallel to the coordinate axes, and two of its corners are $A = (2, 1, 3)$ and $G = (9, 5, 7)$, two of its edges are AE and BF, and two of its faces are $ABCD$ and $EFGH$.
   (a) Find coordinates for the other six vertices;
   (b) Find the lengths AH, AC, AF, FD, and AG;
   (c) Find the distance from G to the xy-plane;
   (d) Find the distance from G to the z-axis;
   (e) Find what C, D, H, and G have in common.
   (f) Is angle FCH a right angle? Explain.
   (g) Find the areas of quadrilaterals $CDEF$ and $CAEG$.
   (h) Show that every pair of interior diagonals (such as FD and CE) bisect each other. (You can do this by either a coordinate geometry method or a synthetic geometry method. Try both approaches.)
   (i) Determine if any pairs of interior diagonals intersect each other perpendicularly. (You can do this by either a coordinate geometry method or a synthetic geometry method. Try both approaches.)
2. An airplane that took off from its airport at noon ($t = 0$ hours) moved according to the formula $(x, y, z) = (15, -20, 0) + t[450, -600, 20]$. What is the meaning of the coordinate 0 in the equation? After twelve minutes, the airplane flew over Bethlehem. Where is the airport in relation to Bethlehem, and how high (in km) was the airplane above the town? What (unrealistic) assumptions did you make in answering this question?
3. Cory is swimming in the pool, according to the equation $C_t = (23 - 2t, 47 - 6t, 92 - 9t)$. Katrina is also swimming in the pool, according to the equation $K_t = (31-2t, 32-3t, 89-6t)$. Determine if the paths of Cory and Katrina intersect or not, and if so, will they collide?
4. (Continuation) Mia and Tiffany are sitting at $P = (1, 25, 4)$ having a chat. Find the position of Cory when the distance between her and Mia and Tiffany is minimum.
5. In a coordinate space, determine if there exists a regular tetrahedron with all of its vertices being lattice points.