Lectures on Challenging Mathematics

Integrated Mathematics 3

Geometry

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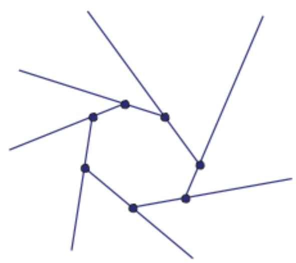
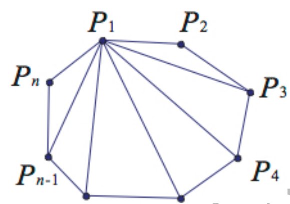
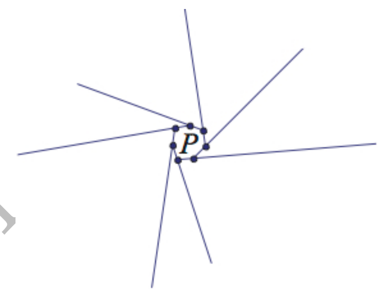
Contents

1.13 Sentry theorem (part 1)

1. Alex walks along the boundary of a n-sided plot of land, writing down the number of degrees turned at each corner. What is the sum of these n numbers for $n = 3, 4, 5, \ldots$? For example, for $n = 7$, the diagram is shown below to the left.







2. *Sentry theorem.* The sum of the exterior angles (one per vertex) of any polygon is 360° degrees. The sum of interior angles of an n-sided convex polygon is $(n-2) \cdot 180°$.

   (a) Polygon $P_1P_2 \ldots P_n$ is triangulated as shown above. Explain how does the statement of the Sentry theory follow from the diagram.

   (b) The sides of a polygon are cyclically extended to form *rays*, creating one exterior angle at each vertex. Viewed from a great distance, how does this figure illustrate the Sentry theorem?

3. A polygon is *equilateral* if its sides have the same length. A polygon is *equiangular* if its interior angles are the same size. For a triangle, equilateral is equivalent to equiangular. For polygons with more than three sides, these two concepts are not equivalent anymore. For example, a rhombus is equilateral but not necessarily equiangular. On the other hand, a rectangle is equiangular but not necessarily equilateral. A polygon that is both equilateral and equiangular is called *regular*. A regular quadrilateral is called a square.

   An equiangular polygon with n sides has 162-degree interior angles. Find the integer n.

4. Let $CHOPIN$ be a regular hexagon, and let $OPERA$ be a regular pentagon. Find all possible values of the measure of $\angle PIE$.

5. In isosceles triangle ABC with $AB = AC$, point D lies on side BC such that $AD = DB$ and $AC = CD$. Compute the angles of triangle ABC.

1.20 Similarity of triangles (part 2)

1. *SAS Similarity theorem.* If an angle of one triangle is congruent to an angle of another triangle and the sides including those angles are in proportion then the triangles are similar.

   In triangle ABC, points D and E lie on the sides AB and AC, respectively. Given that $BD = 8$, $AD = 2$, $AE = 3$, $CE = 12$, $DE = 4$, use SAS Similarity to find the length of BC.

2. (Continuation) Let K and L be the feet of the perpendiculars from A onto DE and BC, respectively.

   (a) Explain why points A, K, and L lie on the same line.

   (b) Find $\frac{AK}{AL}$ and the ratio between the areas of triangles ADE and ABC.

3. (Continuation) Points F and G lie on the sides AB and AC, respectively, such that $BF = 2$ and $CG = 3$. Find the length of segment FG.

   Begin your solution by finding a pair of similar triangles and by writing the similarity statement for all three pairs of corresponding sides.

4. In triangle ABC, points M and N are the midpoints of the sides AB and AC, respectively. Segments BN and CM intersect at P. Find the ratios $\frac{BP}{PN}$ and $\frac{CP}{PM}$.

5. *SSS Similarity theorem.* If the sides of two triangles are in proportion, then the two triangles are similar.

   In acute triangle ABC, point D lies on BC such that $AD \perp BC$. Denote by M and N the midpoints of the sides AB and AC. Use SSS Similarity theorem to prove that triangles DMN and ABC are similar.