Lectures on Challenging Mathematics

Integrated Mathematics 3

Algebra

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Contents

1.9 Long division for polynomials (part 2)

1. Simplify the following expressions:

   (a) $\dfrac{6x^2y}{7} \cdot \dfrac{35xy}{9x^3}$ (b) $\dfrac{x^2}{x+5} \cdot \dfrac{x^2+7x+10}{x}$ (c) $\dfrac{2-x}{x-7} \div \dfrac{x^2+5x-14}{4x-28}$

2. Note that the fraction $\dfrac{1492}{123}$ is improper, and can be converted into the mixed number $12\dfrac{16}{123}$ as shown below on the left-hand side. Likewise, the fraction $\dfrac{x^3+4x^2+9x+2}{x^2+2x+3}$ is an example of an improper fraction, and the *long-division* process illustrated below on the right-hand side can be used to put this fraction into the *mixed form* $x+2+\dfrac{2x-4}{x^2+2x+3}$.

$$\begin{array}{r|r} & 16 \\ \hline 123 & 1492 \\ -) & 1230 \\ \hline & 262 \\ -) & 246 \\ \hline & 16 \end{array} \qquad \begin{array}{r|r} & x+2 \\ \hline x^2+2x+3 & x^3+4x^2+9x+2 \\ -) & x^3+2x^2+3x+0 \\ \hline & 2x^2+6x+2 \\ -) & 2x^2+4x+6 \\ \hline & 2x-4 \end{array}$$

   The long-division process can also deal with zeroes and negative coefficients. For example, we can convert the improper fraction $\dfrac{x^3-3x+2}{x^2+2x-2}$ into the mixed form $x-2+\dfrac{3x-2}{x^2+2x-2}$.

$$\begin{array}{r|r} & x-2 \\ \hline x^2+2x-2 & x^3+0x^2-3x+2 \\ -) & x^3+2x^2-2x+0 \\ \hline & -2x^2-x+2 \\ -) & -2x^2-4x+4 \\ \hline & 3x-2 \end{array}$$

   Use the long division process to express the improper fraction $\dfrac{3x^2-1}{x+2}$ into mixed form.

3. Quadratic polynomial $x^2-13x+m$ is divisible by $x-4$. Find the remainder of this polynomial when divided by $x-5$.

4. Use long division to find an integer k such that x^3-7x+k is divisible by $x+1$. Once you find k, factor the cubic polynomial.

5. Use long division to show that x^2+x+1 divides each of the following polynomials. Explain why you actually do *not* need to do all three long divisions to show the statement.

   (a) x^3-1 (b) x^4-x (c) x^5+x+1

1.15 The second look at quadratics and parabolas (part 3)

1. Consider three consecutive integers, with s denoting the smallest one and m denoting the middle one. Suppose that the sum of the squares of these three integers is equal to 2030.

   (a) Show that $s^2 + 2s - 675 = 0$. Solve this equation and find all three integers. (Is the answer unique?)

   (b) Find an equation for m. Solve this equation.

   (c) The equation $s^2 + 2s - 675 = 0$ in part (a) can be rewritten as $(s^2 + 2s + 1) - 676 = 0$. How does this new form compares to the equation in part (b).

2. Simplify the following expressions:

   (a) $\dfrac{x^2 - 9x - 8}{2x^2 - 18x - 16}$ (b) $\dfrac{14x^2 - 7xy}{2x^2 + 5xy - 3y^2}$

3. Suppose k and m are two positive real numbers. Sketch the graph and find, in terms of k and m, the x-intercepts for the following equations:

   (a) $y = x^2 - mx$ (b) $y = x^2 + mx$
   (c) $y = x^2 - k^2$ (d) $y = x^2 + k^2$

   For each of them devise a quick way to write an equation for the symmetry axis of a parabola.

4. The height h (in feet) above the ground of a baseball depends upon the time t (in seconds) it has been in flight. Cameron takes a mighty swing and hits a blooper whose height is described approximately by the equation $h = 80t - 16t^2$. How long is the ball in the air? The ball reaches its maximum height after how many seconds of flight? What is the maximum height?

5. (Continuation) It takes approximately 0.92 seconds for the ball to reach a height of 60 feet. On its way back down, the ball is again 60 feet above the ground; what is the value of t when this happens? Explain.