Lectures on Challenging Mathematics

Math Challenges 2

Number Sense

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"Cogito ergo Sum – "I think, therefore I am

René Descartes (1596–1650)

"Success is not final, failure is not fatal, it is the courage to continue that counts."

Winston Churchill (1874–1965)

"I can see that without being excited, mathematics can look pointless and cold. The beauty of mathematics only shows itself to more patient followers."

Maryam Mirzakhani (1977–2017)

Contents

1.2 The decimal representation of numbers

1. The *decimal system*, accepted by people a long time ago, uses ten and its powers for representing numbers. This is probably because there are ten fingers on two hands and people started counting by using their fingers. For example, $4618 = 4 \cdot 10^3 + 6 \cdot 10^2 + 1 \cdot 10^1 + 8 \cdot 10^0$. Once we choose to work with powers of 10, we write every positive integer as $\overline{\mathtt{a_m a_{m-1} \ldots a_0}}$, where $\mathtt{a_m}, \mathtt{a_{m-1}}, \ldots, \mathtt{a_1}, \mathtt{a_0}$ are decimal digits from the list $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ and the over bar means that we use a numeral system to represent a number. To make a parallel with our previous example, a four-digit number $\overline{\mathtt{abcd}}$ has value

$$\mathtt{a} \cdot 10^3 + \mathtt{b} \cdot 10^2 + \mathtt{c} \cdot 10^1 + \mathtt{d} \cdot 10^0 = 1000\mathtt{a} + 100\mathtt{b} + 10\mathtt{c} + \mathtt{d},$$

and we have used four variables $\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}$ to represent it. The rightmost digit in the decimal representation is called the *units digit* (or *ones digit*, the second rightmost digit is called the *tens digit*, and so on.

Let m be a 3-digit number. The units digit of m is 3, the tens digit of m is 6, and the hundreds digit of m is 4. Determine the number of 3-digit positive integers n such that there is no carrying required when the two integers m and n are added.

2. (a) Any six-digit number lies in between two consecutive powers of 10. What are the powers?

   (b) How many positive integers x are there such that $3x$ has 3 digits and $4x$ has four digits?

3. Show that an integer is divisible by 4 if and only if the number formed by its tens and units digits (in that order from left to right) is divisible by 4. Start your explanation with:

   > Suppose n is an integer, a is its units digit, b is its tens digit, and its decimal representation is $n = \overline{\cdots \mathtt{cba}}$. We can write n as the sum of ...

   Query: Can you state and show a similar rule to check if an integer is divisible by 8?

4. Let $\mathtt{a}$, $\mathtt{b}$, $\mathtt{c}$ be three (not necessarily distinct) nonzero digits, that is, the possible values of $\mathtt{a}$, $\mathtt{b}$, $\mathtt{c}$ are $1, 2, \ldots, 9$. Given that the positive integer N always divides the sum of six three-digit numbers $\overline{\mathtt{abc}}$, $\overline{\mathtt{acb}}$, $\overline{\mathtt{bac}}$, $\overline{\mathtt{bca}}$, $\overline{\mathtt{cab}}$, $\overline{\mathtt{cba}}$, what is the maximum possible value of N?

5. (a) Find the greatest four-digit number whose product of the digits is equal to 240.

   (b) How many four-digit numbers are there for which the product of the digits is equal to 240?

1.8 Dealing with remainders (part 2)

1. List all possible remainders when the first hundred perfect squares $1^2, 2^2, 3^2, \ldots, 100^2$ are divided by 13. When solving this problem, what assumption(s) did you make?

2. Ryan claims that he solved the following problem:

   > A very large lucky number N consists of eighty-eight 8s in a row. Find the remainder when N is divided by 6.

   Ryan's solution is:

   > The remainder when N is divided by 6 is the same as the remainder when the sum of the digits of N is divided by 6. Because $88 \cdot 8 = 702 + 2$, the remainder when N is divided by 6 is 2.

   (a) Is Ryan's numerical answer correct?

   (b) Is Ryan's reasoning correct?

   (c) Solve the same problem for another lucky number: thirteen 13's in a row.

3. (a) What is the remainder when 2^{2010} is divided by 7?

   (b) What is the remainder when the sum

   $$s = 1! + 1! + 2! + 3! + 5! + 8! + 13! + 21! + 34! + 55! + 89!$$

   is divided by 100?

   (c) Find the remainder when $3^1 \cdot 3^2 \cdot 3^3 \cdots 3^{42} \cdot 3^{43}$ is divided by 11.

4. Giving the fact that $6(1^2 + 2^2 + \cdots + n^2) = 2n^3 + 3n^2 + n$ holds for every positive integer n, find the remainder when the sum $1^2 + 2^2 + 3^2 + \cdots + 50^2$ is divided by 15.

   Query: Explain why the sum $1^2 + 2^2 + \cdots + n^2$ should be divisible by three different primes for large n.

5. When General Han counts the soldiers in his army, he uses the following method. He orders them to line up in rows of 9, then in rows of 10, and finally, in rows of 11, and each time he counts the number of soldiers not in a row. One morning, he finds that there are 7 soldiers left when the rest are in rows of 9, 5 soldiers left when the rest are in rows of 10, and 9 soldiers left when the rest are in rows of 11. He knows that there are 1000 soldiers in his army. How many of the soldiers are present this morning?