Lectures on Challenging Mathematics

Math Challenges 2

Geometry

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"Cogito ergo Sum" – "I think, therefore I am"
René Descartes (1596–1650)

"Success is not final, failure is not fatal, it is the courage to continue that counts."
Winston Churchill (1874–1965)

"I can see that without being excited, mathematics can look pointless and cold. The beauty of mathematics only shows itself to more patient followers.
Maryam Mirzakhani (1977–2017)

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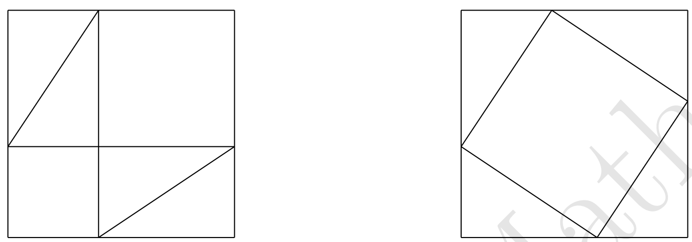
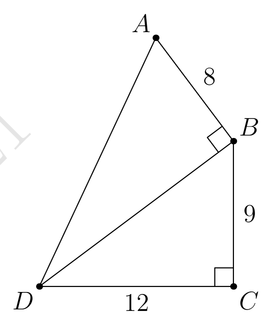
1.5 The Pythagorean Theorem

1. The *Pythagorean Theorem* states that in a right triangle the square of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the other two sides.

   The two-part diagram below, which shows two different dissections of the same square, was designed to help prove the Pythagorean Theorem. Provide the missing details.





2. In baseball, the bases are placed at the corners of a square whose sides are 90 feet long. Home plate and second base are at opposite corners. How far is it from home plate to second base?

3. In quadrilateral $ABCD$ (shown at right), $\angle ABD = \angle BCD = 90^\circ$. Given that $AB = 8$, $BC = 9$, and $CD = 12$, find the length of AD.

4. A 25-foot ladder reaches 24 feet up the side of building. Then the top of the ladder slides down 4 feet. How many additional feet does the bottom of the ladder slide out from the base of the building?

5. Two right triangles of different shape with side lengths 3 and 4 can be put together to form a triangle. There is only one way to do so. Find the side lengths of this triangle.

   We can also put these two triangles together to form a quadrilateral. Find three different quadrilaterals that can be constructed this way.

1.10 Congruence of triangles (part 2)

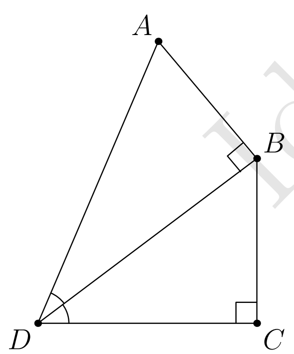
1. Two triangles are congruent if they satisfy the *Angle-Side-Angle (ASA) Conditions*:

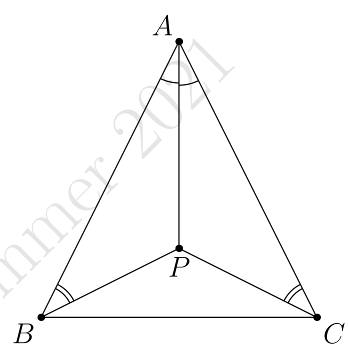
   If two angles and the included side of one triangle are congruent to two angles and the included side of another triangle, then the triangles are congruent.

   Let $ABCD$ be a quadrilateral with four right angles. Draw diagonal AC. Explain why triangles ABC and CDA are congruent. Conclude that in this quadrilateral, the opposite sides are of the same length. What is the name of this quadrilateral?

2. (Continuation) Draw diagonal BD. Explain why the diagonals in such quadrilateral are of the same length by spotting two congruent triangles. Specify which congruence theorem(s) you are using.

3. Suppose we have two right triangles ABD and BCD with $\angle ABD = \angle BCD = 90^\circ$ and $\angle ADB = \angle CDB$. The diagram is shown below to the left. Chris says that using ASA congruence, we should get that $\triangle ABD \cong \triangle BCD$. Do you agree with his argument? Explain.





4. Suppose inside the triangle ABC there is a point P such that $\angle PBA = \angle PCA$ and $\angle PAB = \angle PAC$. The diagram is shown above to the right. Show that triangle ABC is isosceles by completing the following steps of the proof.

   The sum of the angles in a triangle is 180°. We have

   $\angle PBA + \angle PAB +$ ________ $= 180^\circ$ and $\angle PCA + \angle PAC + \angle APC =$ ________.

   Therefore __________ $= \angle APC$. Note that

   A: ______ $= \angle APC$;
   S: ______ $=$ ______;
   A: $\angle BAP =$ ______.

   By ________ congruence, triangle __________ is ______________ to triangle ________, hence $AB =$ ________, and therefore triangle ABC is ________.

5. The *distance* between a point P and a line ℓ is just the length of the segment PQ, where point Q lies on ℓ and PQ is perpendicular to the line ℓ. Explain why this is the shortest straight path from P to ℓ.

   Show that any point P on the angle bisector of $\angle BAC$ is equidistant (lies at the same distance) from the two sides of $\angle BAC$.