Lectures on Challenging Mathematics

Integrated Mathematics 1

Algebra (Part 2)

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3.7 Factorization of quadratic polynomials

1. The task of the section is to learn how to factor given quadratic polynomial; that is, express a quadratic polynomial as a product of nonconstant linear polynomials with integer coefficients. Let's start with confirming the expansion:

$$(x+p)(x+q) = x^2 + (p+q)x + pq.$$

If quadratic polynomial $x^2 + bx + c$ can be factored into the product $(x+p)(x+q)$ (that is, $x^2 + bx + c = (x+p)(x+q)$), then what are b and c, in terms of p and q?

2. In this problem, we deal with the factorization of quadratic polynomials in general form. For example, we want to find polynomials $(ax+b)(cx+d) = 6x^2 + 17x + 12$. We have, by expanding, the following form:

$$\begin{array}{rr}
 & ax + b \\
\times) & cx + d \\
\hline
 & bd \\
 & (ad)x \\
 & (bc)x \\
+) & (ac)x^2 \\
\hline
 & 6x^2 + 17x + 12
\end{array}$$

Because $ac = 6$, the possible choices for $\{a, c\}$ are $\{\pm1, \pm6\}$ and $\{\pm2, \pm3\}$. Because $bd = 12$, the possible choices for $\{b, d\}$ are $\{\pm1, \pm12\}$, $\{\pm2, \pm6\}$, $\{\pm3, \pm4\}$. By checking with the condition $ad + bc = 17$, it is not difficult to find that $(a, b, c, d) = (2, 3, 3, 4)$.

Repeat the above the process to factor $2x^2 - 3x - 20$.

3. For each of the following polynomials, determine if it can be factored. If *yes*, factor it; if *no*, state that they cannot be factored.

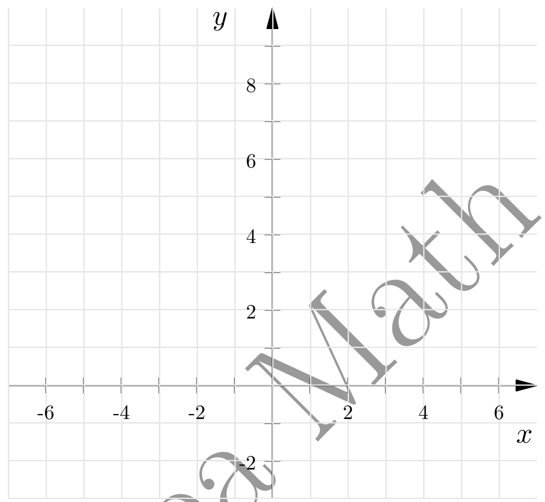
(a) $20x^2 - 45$
(b) $x^2 - 14x + 45$
(c) $x^2 + 5x - 7$
(d) $2x^2 + 9x - 5$
(e) $3x^2 + 5x + 2$
(f) $2x^2 + 4x - 3$

4. The product of two consecutive odd positive integers is equal to 4095. Set the numbers to be $2n - 1$ and $2n + 1$. Set up an equation for n and solve the equation to find the two numbers.

5. Factor $ax + bx - ay - by$.

4.8 Solving quadratic equations (part 1)

1. Graph the quadratic functions $y = x^2$, $y = (x-2)^2$, and $y = (x+3)^2$. Describe in general terms how the graph of $y = x^2$ is transformed to produce the graph of $y = (x-h)^2$.



2. (Continuation) You have seen that the graph of any quadratic function is a parabola that is symmetrical with respect to a line called the *axis of symmetry*, and that each such parabola also has a lowest or highest point called the *vertex*. For each of the graphs identify the coordinates of the vertex and write an equation for axis of symmetry.

3. When asked to solve the equation $(x-3)^2 = 2$, Jess said, "That's easy - just take the square root of both sides." What are the two possible values for x, in exact form? Plot the graph of $y = (x-3)^2$ and explain the meaning of solving equation $(x-3)^2 = 2$.

4. (Continuation) However, when asked to solve the equation $x^2 - 8x = 1$, Jess said, "Hmm . . . not so easy, but I think that adding something to both sides of the equation is the thing to do." This is indeed a good idea, but what number should Jess add to both sides?

5. Use the method above to solve quadratic equations:

   (a) $x^2 - 4x = 16$ (b) $x^2 + 10x = 144$
   (c) $x^2 - 14x - 1 = 0$ (d) $x^2 + 6x + 16 = 0$

5.7 Sequences, series, growth, and decay (part 3)

1. Find all possible values of x such that $(a_1, a_2, a_3, \dots) = (3, 7, 11, \dots, x, 55, \dots)$ is an arithmetic progression.

   Note that we call 7 the *arithmetic mean* of 3 and 11 because $7 = (3 + 11)/2$. Find the arithmetic mean of each of the following pairs of terms:

   (a) a_{11} and a_{13} (b) a_2 and a_{100} (c) a_m and a_n

2. Three-term sequence $(a_1, a_2, a_3) = (8, 5, 2)$ satisfies the following conditions: It is arithmetic and all its terms are different digits. How many 3-term sequences satisfy these two conditions?

3. The *exponential function* D defined by $D(n) = 2500(1.36)^n$ describes the number of mold spores found growing on a piece of fruit n days after the mold was first discovered.

   (a) How many spores were on the fruit when the mold was first discovered?

   (b) How many spores were on the fruit two days before the mold was first discovered?

   (c) What is the daily growth rate of this population?

4. (Continuation) Plot points $(n, D(n))$ for $n = 0, 1, 2, 3, 4, 5$ in the coordinate plane.

   (a) Do these points lie on a straight line?

   (b) Sketch a curve passing through these points.

   (c) Use the graphing device to graph the curve $y = 2500(1.36)^x$ and compare this curve with the curve your drew in part (b).

   (d) There is a point on the curve with x-coordinate equal to 0.5. Plot this point. Estimate the value of the y-coordinate of this point. What is the meaning of this value? Repeat this process for $x = 3.6$.

5. (Continuation) There is a point on the curve with x-coordinate equal to -1. Plot this point. Estimate the value of the y-coordinate of this point. What is the meaning of this value? Repeat this process for $x = -2$ and $x = -4.5$.

   The graph of $y = 2500(1.36)^x$ indicates that there is an output (y value) for every real number input x. We say that the *domain* of the function $y = 2500(1.36)^x$ is all real numbers. What is the range of this function; that is, what are all the possible values of the output y?