Lectures on Challenging Mathematics

Math Challenges 1

Number Sense

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“Cogito ergo Sum – “I think, therefore I am

René Descartes (1596–1650)

“Success is not final, failure is not fatal, it is the courage to continue that counts.”

Winston Churchill (1874–1965)

“I can see that without being excited, mathematics can look pointless and cold. The beauty of mathematics only shows itself to more patient followers.”

Maryam Mirzakhani (1977–2017)

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1.3 Basic number sense practices (part 2)

1. A *composite* number is a positive integer greater than 1 that has at least one divisor other than 1 and itself. What is the least positive integer divisible by four different composite numbers?

2. How many three-digit positive integers are there? How many of them are divisible by 3? How many of them have sum of the digits equal to 3?

3. Little John bought two tickets to a movie for him and his sweetheart. Sadly, it turned out that the two seats they got are not next one another, but there are eleven empty seats between them! During each commercial John can move two or four seats closer to his sweetheart. Can he eventually sit next to her?

4. If n^2 is the greatest perfect square that divides 12!, what is n?

5. What is the greatest prime factor of

   (a) $8 \cdot 9 \cdot 10 + 18 \cdot 19 \cdot 20$

   (b) $15^{10} + 15^{11} + 15^{12} + 15^{13}$

1.7 Quotients and remainders (part 2)

1. Every five months, Hal has to replace the batteries in his calculator. He changed them the first time in May. In what month will they be changed the 25^{th} time?

2. What is the least positive integer that has a remainder of 1 when divided by 2, a remainder of 2 when divided by 3, and a remainder of 4 when divided by 5? What is the second least positive integer satisfying these conditions?

3. Find the remainders when 525, 816, and $525 + 816$ are divided by 14. Think of a pair of three-digit integers a and b, and a two-digit integer divisor n that no one in the class will think of. Find the remainders when a, b, and $a + b$ are divided by n. Make an observation and complete the following sentence:

   The remainder of the sum of two integers when divided by an integer n is equal to ____ divided by an integer n.

4. One can find the remainder when the sum

$$2 + 9 + 16 + 23 + \cdots + 702$$

   is divided by 7 with or without finding the actual sum. Do so in both ways.

5. Russell thinks of an integer number. He tells you that if he adds 100 to his number the remainder of the new number when divided by 19 is 2. What would be the remainder if he were to multiply his number by 100 instead and then divide it by 19?

1.10 Switching the order of operations (part 3)

1. Evaluate each of $\dfrac{1}{2\cdot 3\cdot 4}+\dfrac{1}{3\cdot 4\cdot 5}$ and $\left(\dfrac{1}{2\cdot 3\cdot 4\cdot 5}+\dfrac{1}{3\cdot 4\cdot 5\cdot 6}\right)\cdot 84$.

2. Given that m and n are positive integers such that

$$2\times 2\times 4\times 4\times 4\times 4\times \underbrace{8\times 8\times\cdots\times 8}_{\text{eight 8s}}\times\underbrace{5\times 5\times\cdots\times 5}_{m\text{ 5s}}=1\underbrace{0\ldots 0}_{n\text{ 0s}},$$

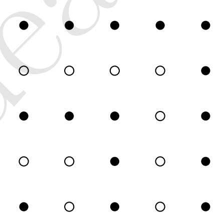
find m and n.

3. The product of all positive integer divisors of 72, including 1 and 72 itself, is equal to 2^k3^ℓ. Find the integers k and ℓ.

4. Compute each of the following sums.

(a) $1+3+5+\cdots+21$ (b) $1+3+5+\cdots+63$ (c) $1+3+5+\cdots+99$

What is common to all the answers that you get? Do you think this is a coincidence? The diagram below may be helpful in explaining your result.



5. Evaluate each of $(123+4)(123+5)-123\cdot 132$ and $(9876+4)(9876+5)-9876\cdot 9885$.