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1. Let $p$ be an odd prime, and let $a$ and $b$ be two relatively prime positive integers. Find the set of all possible values of

$$\gcd(a + b, \frac{a^p + b^p}{a + b}).$$

2. Let $S(x)$ be the sum of the digits of the its decimal representation.

   (a) Prove that for every positive integer $x$, \( \frac{S(x)}{S(2x)} \leq 5 \). Can this bound be improved?

   (b) Prove that \( \frac{S(x)}{S(3x)} \) is not bounded.

3. Call a number **very composite** if it has at least 2008 distinct prime factors. Do there exist 2008 consecutive very composite numbers?

4. Tanya chooses a positive integer $x \leq 100$, and Sasha is trying to guess this number. She can select two positive integers $m$ and $n$ less than 100 and ask for the value of $\gcd(x + m, n)$. Show that Sasha can determine Tanya’s number with at most seven questions (the numbers $m$ and $n$ can change each question).

5. Find all primes $p$ such that $(p - 1)! + 1$ is a perfect power of $p$. 
