Lectures on Challenging Mathematics

Math Olympiads

Number Theory

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1.15 Proofs in modular arithmetic (part 2)

1. Let p be an odd prime, and let a and b be two relatively prime positive integers. Find the set of all possible values of
$$\gcd\left(a+b, \frac{a^p+b^p}{a+b}\right).$$

2. Let $S(x)$ be the sum of the digits of the its decimal representation.

   (a) Prove that for every positive integer x, $\frac{S(x)}{S(2x)} \le 5$. Can this bound be improved?

   (b) Prove that $\frac{S(x)}{S(3x)}$ is not bounded.

3. Call a number *very composite* if it has at least 2008 distinct prime factors. Do there exist 2008 consecutive very composite numbers?

4. Tanya chooses a positive integer $x \le 100$, and Sasha is trying to guess this number. She can select two positive integers m and n less than 100 and ask for the value of $\gcd(x+m, n)$. Show that Sasha can determine Tanya's number with at most seven questions (the numbers m and n can change each question).

5. Find all primes p such that $(p-1)!+1$ is a perfect power of p.