Lectures on Challenging Mathematics

Math Olympiads

Combinatorics

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1. Determine the number of permutations π of $(1, 2, \ldots, 10)$ satisfying the following property:

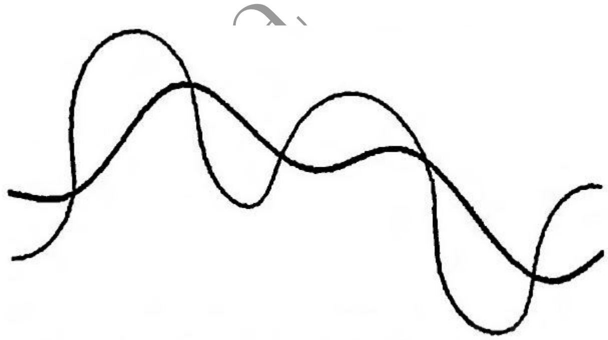
   (a) For every $1 \le i \le 10$, $\pi^{(30)}(i) = i$;

   (b) For every $1 \le k \le 29$, there is a $1 \le i \le 10$ such that $\pi^{(k)}(i) \ne i$.

   (For $n \ge 1$, $\pi^{(n+1)}(i) = \pi(\pi^{(n)}(i))$.)

2. At a conference, each participant is acquainted with at least one other participant, and for each pair of participants there is a participant not acquainted to both. Prove that one can divide all the participants into three nonempty groups so that each participant would be acquainted with at least one person in the group the participant is assigned to.

3. Here is a wonderful problem from the book *Which Way Did the Bicycle Go?* (by J. Konhauser, D. Velleman, S. Wagon, MAA 1996): Imagine that while walking down a dirt path (shown below), you see the following pair of tire tracks that you guess were made by a bicycle. Can you determine which way the bicycle was going?



4. Fifteen rooks are placed on a regular chessboard satisfying the following conditions: each unit field is occupied by at most one rook and there is at least one rook in each row and each column. Prove that one can remove one of the rooks so that the remaining rooks still satisfy the above conditions.

   Do you see the relation between this problem and the following problem?

   > Find the smallest positive integer n such that if n unit squares of a 1000×1000 unit-square board are colored, then there will exist three colored unit squares whose centers form a right triangle with legs parallel to the edges of the board.

5. Determine if there is a positive integer n such that the four most significant digits of n are 2015; that is, $n! = 2015\ldots$.