

# Lectures on Challenging Mathematics

## Math Olympiads

### Algebra

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## 1.18 Introduction to functional equations (part 3)

1. Find all functions  $f: \mathbb{R} \rightarrow \mathbb{R}$  such that  $f(2x) = 2f(x)$  and  $|x - f(x)| \leq 1$  for all  $x$  in  $\mathbb{R}$ .
2. Identify all asymptotic behavior of the graph

$$g(x) = \frac{x^4 - x^3 + 7x^2 - x + 1}{x^3 + x}.$$

Determine, with justification, if this graph has a half-turn symmetry.

3. Determine if the following statement is true.

If  $f: \mathbb{R} \rightarrow \mathbb{R}$  is a function for which  $f(x + 10) = f(x) + f(10)$ , then  $f$  is linear.

4. Let  $f(x)$  be a quadratic polynomial with real coefficients. Prove that  $f(x)$  cannot be written as a sum of two periodic functions.
5. Call a real-valued function  $f$  *very convex* if

$$\frac{f(x) + f(y)}{2} \geq f\left(\frac{x+y}{2}\right) + |x - y|$$

holds for all real numbers  $x$  and  $y$ . Prove that no very convex function exists.