Lectures on Challenging Mathematics

Elements of Math Olympiads

Geometry

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Zuming Feng
Phillips Exeter Academy and IDEA Math
zfeng@exeter.edu

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1.10 Angle chasing and the centers of triangles (part 3)

1. Let ABC be an acute triangle. Point P lies on segment AC and point A lies on segment BQ such that triangle BPQ is similar to triangle ABC. Prove that the circumcenter of triangle ABC is the orthocenter of triangle BPQ.

2. The incircle of triangle ABC touches sides AB and AC at Q and P, respectively. The bisectors of angles B and C meet line PQ at X and Y, respectively. Prove that $BCXY$ is cyclic and determine its circumcenter.

3. Let $ABCD$ be a convex quadrilateral with $\angle ABC = \angle CDA$. The circumcircle of triangle ACD meets line segment BC at X and the circumcircle of triangle ABC meets line segment CD at Y. Prove that $BY = DX$.

4. In triangle ABC, H is the orthocenter and O is the circumcenter. Denote by H_a the midpoint of AH and by M_a the midpoint of BC.

   (a) Prove that HH_aOM_a is a parallelogram.

   (b) Similarly, we define points H_b, H_c, M_b, and M_c. Show that H_aM_a, H_bM_b, H_cM_c are concurrent.

5. Points D and E lie on side AC of triangle ABC. Given that $\angle C = 40°$, $\angle ABD = 10°$, $\angle ABE = 40°$, and $\angle ABC = 50°$. Show that $CE = 2AD$ by

   (a) establishing the fact that the circumcircle of triangle ABD passes through the midpoint of side BC;

   (b) applying a proper reflection.