Lectures on Challenging Mathematics

Introduction to Math Olympiads

Geometry

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"Cogito ergo Sum" – "I think, therefore I am"

René Descartes (1596–1650)

"Success is not final, failure is not fatal, it is the courage to continue that counts."

Winston Churchill (1874–1965)

"I can see that without being excited, mathematics can look pointless and cold. The beauty of mathematics only shows itself to more patient followers."

Maryam Mirzakhani (1977–2017)

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1.4 Angle chasing and centers of triangles (part 2)

1. Let ABC be an acute triangle. Point P lies on segment AC and point A lies on segment BQ such that triangle BPQ is similar to triangle ABC. Prove that the circumcenter of triangle ABC is the orthocenter of triangle BPQ.

2. The incircle of triangle ABC touches sides AB and AC at Q and P, respectively. The bisectors of angles B and C meet line PQ at X and Y, respectively. Prove that $BCXY$ is cyclic and determine its circumcenter.

3. Let $ABCD$ be a convex quadrilateral with $\angle ABC = \angle CDA$. The circumcircle of triangle ACD meets line segment BC at X and the circumcircle of triangle ABC meets line segment CD at Y. Prove that $BY = DX$.

4. In triangle ABC, H is the orthocenter and O is the circumcenter. Denote by H_a the midpoint of AH and by M_a the midpoint of BC.

   (a) Prove that HH_aOM_a is a parallelogram.

   (b) Similarly, we define points H_b, H_c, M_b, and M_c. Show that H_aM_a, H_bM_b, H_cM_c are concurrent.

5. Points D and E lie on side AC of triangle ABC. Given that $\angle C = 40°$, $\angle ABD = 10°$, $\angle ABE = 40°$, and $\angle ABC = 50°$. Show that $CE = 2AD$ by

   (a) establishing the fact that the circumcircle of triangle ABD passes through the midpoint of side BC;

   (b) applying a proper reflection.

1.12 Introduction to Simson line and Miquel's theorem

1. Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively.

   (a) Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.

   (b) Note that it seems that PQ, RS, AB are concurrent. Is it true? Maybe the next problem can tell us why.

2. (Simson line) Consider point P on the circumcircle of triangle ABC. Let points D, E, and F be the feet of the of the perpendiculars from P to lines AB, AC, and BC, respectively. Prove that D, E, and F are collinear. The line through these points is called the *Simson line* of point P with respect to triangle ABC.

   State the converse statement and determine if the converse is true.

3. (Miquel's theorem) Let ABC be a triangle. Points X, Y, and Z lie on sides BC, CA, and AB, respectively. The circumcircles of triangles AYZ, BZX, and CXY meet at a common point – the *Miquel point*. (Indeed, X, Y, Z can lie on lines AB, BC, CA.)

   State the converse statement and determine if the converse is true.

4. (Continuation) Prove that the circumcenters of triangles AYZ, BZX, and CXY form a triangle similar to triangle ABC.

5. [Miquel's theorem] Consider quadrilateral $ABCD$ and suppose lines AB and CD intersect in point E and lines BC and AD intersect in point F. Prove that the circumcircles of triangles ADE, BCE, CDF, and ABF (*Miquel circles*) intersect at one point, called (*Miquel point*). What is the necessary and sufficient condition for the Miquel point to lie on the diagonal EF? The existence of the Miquel point can be established in at least two different approaches, one by angle chasing and one by Simson line. Please try both methods.