Lectures on Challenging Mathematics

Elements of Math Olympiads

Combinatorics

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1.13 Mathematical arguments – The well ordering principle

1. *The well ordering principle*: We can arrange a finite set of discrete values in order. (It is simply the naive version of the Zorn's lemma in set theory.) You might think it is just common sense, right? Well, it is a very useful common sense indeed. Let's see an example.

   There are 25 people sitting around a table, and each person has two cards. One of the numbers $1, 2, \ldots, 25$ is written on each card, and each number occurs on exactly two cards. At a signal, each person passes one of his cards, the one with the smaller number, to his right-hand neighbor. Prove that, sooner or later, one of the players will have two cards with the same number.

2. There are n points in a plane. Any three of the points form a triangle of area less than or equal to 1. Show that all n points lie in a triangle of area less than or equal to 4.

3. Let S be a set of finitely many points on a plane such that any three distinct points in S form a right triangle. Determine with proof the maximum number of elements in S.

4. There are 99 points in a plane such that no three are collinear and no four lie on a circle. Prove that it is always possible to choose three of the points and draw a circle through these points, such that exactly 48 of the remaining points lie inside this circle.

5. Consider a network of finitely many balls, some of which are joined to one another by wires. We shall color the balls black and white, and call a network *diverse* if each white ball has at least as many as black as white neighbors, and vice versa. Given any network, is there a coloring that makes it diverse?