

Lectures on Challenging Mathematics

Introduction to Math Olympiads

Combinatorics

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“Cogito ergo Sum” – “I think, therefore I am”

René Descartes (1596–1650)

“Success is not final, failure is not fatal, it is the courage to continue that counts.”

Winston Churchill (1874–1965)

“I can see that without being excited, mathematics can look pointless and cold. The beauty of mathematics only shows itself to more patient followers.”

Maryam Mirzakhani (1977–2017)

Contents

1	Math Olympiads 2, Combinatorics	3
1.1	Parity argument	3
1.2	A touch on the Vandermonde's identity	4
1.3	Should I gamble? No, you should not!	5
1.4	General terminology in graph theory (part 1)	6
1.5	Expected value (part 1)	7
1.6	General terminology in graph theory (part 2)	8
1.7	Counting with linear arrangements and grids (part 1)	9
1.8	General terminology in graph theory (part 3)	10
1.9	Expected value (part 2)	12
1.10	Connectivity of graphs (part 1)	13
1.11	Recursive counting (part 2)	14
1.12	Connectivity of graphs (part 2)	16
1.13	The well ordering principle	17
1.14	Eulerian walks and Hamiltonian cycles	18
1.15	Circular arrangements	20
1.16	Trees (part 1)	21
1.17	Coloring and assigning weights (part 1)	22
1.18	Trees (part 2)	23
1.19	Coloring and assigning weights (part 2)	24
1.20	Graph theory in action	25
2	Math Olympiads 2, Combinatorics Challenges	27
2.1	Counting with linear arrangements and grids (part 2)	27
2.2	Recursive counting (part 3)	29
2.3	Challenges in enumerative counting (part 1)	30
2.4	Challenges in enumerative counting (part 2)	31
2.5	Math reasoning gems from AIME 2016	32

1.5 Expected value (part 1)

1. *Expected value* is the average value of a variable whose values are randomly determined by a probability experiment (the number of aces when three dice are tossed, for example). If you think this definition confusing, do not worry, you are not alone. Indeed, let's use some common sense to understand the meaning of this math concept.
 - (a) A five-step random walk starts at the origin, and each step is either one unit to the right or one unit to the left. If this five-step walk is to be performed 1 million times, what is your prediction of the average of all the final positions?
 - (b) A standard six-sided die is to be rolled 3000 times. Predict the average result of all these rolls. The correct answer to this question is called the *expected value* of rolling a die. What is the expected *sum* of two dice? of ten dice? of n dice?
2. In the *Tri-State Megabucks Lottery*, a player chooses six different numbers from 1 to 40, hoping to match all six numbers to be randomly drawn that week. The order in which the numbers are drawn is unimportant. What is the probability of winning this lottery? The jackpot is ten million dollars (ten *megabucks*). A fair price to pay for a ticket is its *expected value*. Why? Calculate this value, which is more than \$2. Assume that there is a unique winner (the winning ticket was sold only once).
3. The expected value need not be a value of the variable—the expected number of heads is 2.5 when five coins are tossed. Indeed, the expected value is calculated by considering every possible outcome of the experiment. It may be expressed as

$$p_1v_1 + p_2v_2 + p_3v_3 + \cdots + p_nv_n,$$

in which each value v_k has been multiplied by the probability p_k that v_k will occur.

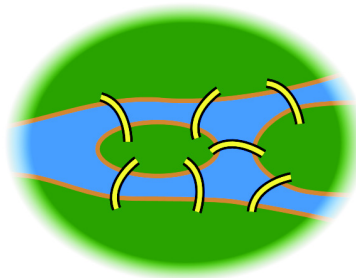
Before beginning a one-dimensional, five-step random walk, what would be your prediction for the distance from the origin to the final position?

4. Let $P_1P_2P_3P_4P_5P_6P_7P_8$ be a convex octagon. An integer i is chosen uniformly at random from 1 to 7, inclusive. For each vertex of the octagon, the line between that vertex and the vertex i vertices to the right is painted red. What is the expected number times two red lines intersect at a point that is not one of the vertices, given that no three diagonals are concurrent?
5. Two jokers are added to a 52 card deck and the entire stack of 54 cards is shuffled randomly. Find expected number of cards that will be strictly between the two jokers, trying *two* different approaches to the problem. One method uses a certain summation, one does not.

1.14 Eulerian walks and Hamiltonian cycles

1. The Seven Bridges of Königsberg problem is a historically notable problem in mathematics. Its negative resolution by Leonhard Euler in 1735 laid the foundations of graph theory and prefigured the idea of topology.

The city of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River, and included two large islands which were connected to each other and the mainland by seven bridges. See the figure shown below.



The problem was to find a walk through the city that would cross each bridge once and only once. The islands could not be reached by any route other than the bridges, and every bridge must have been crossed completely every time; one could not walk halfway onto the bridge and then turn around and later cross the other half from the other side. The walk need not start and end at the same spot. Euler proved that the problem has no solution. There could be no non-retracing the bridges. The difficulty was the development of a technique of analysis and of subsequent tests that established this assertion with mathematical rigor.

Reformulate the above problem by graph theory terminologies.

2. In graph theory, an *Eulerian walk* is a walk in a graph which visits every edge exactly once. Similarly, an *Eulerian cycle* (or *Eulerian circuit*) is an Eulerian walk which starts and ends on the same vertex. They were first discussed by Leonhard Euler while solving the famous Seven Bridges of Königsberg problem in 1736. Euler proved that a necessary condition for the existence of Eulerian cycles is that all vertices in the graph have an even degree. (Hence the answer of the Königsberg problem is negative.) Now it is your turn to prove this result. More precisely, show that if a connected graph has more than two nodes with odd degree, then it has no Eulerian walk.

Euler stated without proof that this condition is also sufficient; that is, a connected graph with all vertices of even degree has an Eulerian circuit. The first complete proof of this latter claim was published posthumously in 1873 by Carl Hierholzer. The term Eulerian graph has two common meanings in graph theory. One meaning is a graph with an Eulerian circuit, and the other is a graph with every vertex of even degree. These definitions are equivalent to each other for connected graphs. In the next few problems, we establish this equivalence relation.

3. Let G be a connected graph and every vertex in G has even degree. Let v be a fixed vertex in G . Consider the set of all closed walks starting and ending at v that uses every edge at most once. Suppose that this set is nonempty, and let W be such a walk in this set with the maximum number of edges. Further assume that there is an edge not belonging to W . Prove that there is an edge not belonging to W that is adjacent to a vertex in W .
4. Show that there is an Eulerian walk in a given connected graph with no odd-degree vertices, and every Eulerian walk in this graph is an Eulerian cycle.
5. A *Hamiltonian path* is a path in an undirected graph that visits each vertex exactly once. A *Hamiltonian cycle* (or *Hamiltonian circuit*) is a Hamiltonian path that is a cycle.

A *Hamiltonian path* or *traceable path* is a path that visits each vertex exactly once. A graph that contains a Hamiltonian path is called a *traceable graph*. A *Hamiltonian cycle*, or *Hamiltonian circuit*, or *vertex tour* is a cycle that visits each vertex exactly once (except the vertex that is both the start and end, and so is visited twice). A graph that contains a Hamiltonian cycle is called a *Hamiltonian graph*.

Hamiltonian paths and cycles and cycle paths are named after William Rowan Hamilton who invented the Icosian game, now also known as Hamilton's puzzle, which involves finding a Hamiltonian cycle in the edge graph of the dodecahedron. See the figure shown below. Now it is your turn to find it. Before you do that, are you convinced that this is the planar graph of a dodecahedron?

