

Lectures on Challenging Mathematics

Elements of Math Olympiads

Algebra

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Internal Use

1.8 Basic properties of polynomials

1. [Lagrange's Interpolation Formula] There is a unique second degree polynomial $p(x)$ passing through points $(1, 5)$, $(3, 8)$, $(6, -7)$. Explain why

$$p(x) = \frac{5(x-3)(x-6)}{(1-3)(1-6)} + \frac{8(x-1)(x-6)}{(3-1)(3-6)} - \frac{7(x-1)(x-3)}{(6-1)(6-3)}.$$

Find a third degree polynomial that passes through points $(1, 0)$, $(2, 1)$, $(4, 14)$, and $(6, 55)$.

2. (Continuation) In general, let x_0, x_1, \dots, x_n be distinct real numbers, and let y_0, y_1, \dots, y_n be arbitrary real numbers. Then there exists a unique polynomial $P(x)$ of degree at most n such that $P(x_i) = y_i$, $i = 0, 1, \dots, n$. Show that this polynomial is

$$P(x) = \sum_{i=0}^n y_i \frac{(x-x_0)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}.$$

3. Let $P(x)$ be a polynomial with leading coefficient 1 and integer coefficients. If u and v are positive integers, where v is not a perfect square, and $u + \sqrt{v}$ is a root of $P(x)$, show that $u - \sqrt{v}$ is also a root of $P(x)$.
4. Let $f(x) = x^4 - 49x^2 - 14x - 1$ and let $g(x) = ax + b$. Find positive integers a and b for which $f(g(x))$ is divisible by $x^2 + 9x + 19$.
5. The polynomial P is a quadratic with integer coefficients. For every positive integer n , the integers $P(n)$ and $P(P(n))$ are relatively prime to n . If $P(3) = 89$, determine with justification the value of $P(10)$.