Lectures on Challenging Mathematics

Introduction to Math Olympiads

Algebra

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"Cogito ergo Sum" – "I think, therefore I am"

René Descartes (1596–1650)

"Success is not final, failure is not fatal, it is the courage to continue that counts."

Winston Churchill (1874–1965)

"I can see that without being excited, mathematics can look pointless and cold. The beauty of mathematics only shows itself to more patient followers."

Maryam Mirzakhani (1977–2017)

Contents

1.8 Polynomials, roots, and coefficients (part 1)

1. [Lagrange's Interpolation Formula] There is a unique second degree polynomial $p(x)$ passing through points $(1,5),(3,8),(6,-7)$. Explain why

$$p(x) = \frac{5(x-3)(x-6)}{(1-3)(1-6)} + \frac{8(x-1)(x-6)}{(3-1)(3-6)} - \frac{7(x-1)(x-3)}{(6-1)(6-3)}.$$

   Find a third degree polynomial that passes through points $(1,0)$, $(2,1)$, $(4,14)$, and $(6,55)$.

2. (Continuation) In general, let $x_0, x_1, \ldots, x_n$ be distinct real numbers, and let $y_0, y_1, \ldots, y_n$ be arbitrary real numbers. Then there exists a unique polynomial $P(x)$ of degree at most n such that $P(x_i) = y_i$, $i = 0, 1, \ldots, n$. Show that this polynomial is

$$P(x) = \sum_{i=0}^{n} y_i \frac{(x-x_0)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}.$$

3. Let $P(x)$ be a polynomial with leading coefficient 1 and integer coefficients. If u and v are positive integers, where v is not a perfect square, and $u+\sqrt{v}$ is a root of $P(x)$, show that $u-\sqrt{v}$ is also a root of $P(x)$.

4. Let $f(x) = x^4 - 49x^2 - 14x - 1$ and let $g(x) = ax + b$. Find positive integers a and b for which $f(g(x))$ is divisible by $x^2 + 9x + 19$.

5. The polynomial P is a quadratic with integer coefficients. For every positive integer n, the integers $P(n)$ and $P(P(n))$ are relatively prime to n. If $P(3) = 89$, determine with justification the value of $P(10)$.

1.13 Introduction to functional equations (part 2)

1. Let $f(x) = x^2 + ax + b$. Show that

$$\frac{1}{2}f(x_1) + \frac{1}{2}f(x_2) \geq f\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right)$$

   for all real numbers x_1 and x_2. When does the equality hold?

   More generally, show that

$$tf(x_1) + (1-t)f(x_2) \geq f(tx_1 + (1-t)x_2)$$

   for all real numbers x_1, x_2 and positive real number t in the interval $[0, 1]$. This is equivalent to saying that $f(x)$ is *convex*; that is, its graph looks like a bowl holding water. Sketch the graph of $y = f(x)$ and explain the terminology.

2. Given a function f for which

$$f(x) = f(398 - x) = f(2158 - x) = f(3214 - x)$$

   holds for all real x, what is the largest number of different values of f that can appear in the list $f(0)$, $f(1)$, $f(2)$, ..., $f(999)$?

3. Prove that the equation $f(g(h(x))) = 0$, where f, g, h are quadratic polynomials can't have solutions $1, 2, 3, 4, 5, 6, 7$, and 8.

   Start your solution by assuming that such polynomials exist.

   (a) How many different values can be in the list $h(1), h(2), \ldots, h(8)$?

   (b) How many different values can be in the list $g(h(1)), g(h(2)), \ldots, g(h(8))$?

   (c) What conclusion(s) do you make?

4. (Continuation) Find quadratic polynomials f, g, h and distinct integers $a_1, a_2, \ldots, a_8$ such that $f(g(h(a_i))) = 0$ where $1 \leq i \leq 8$.

5. Let $\mathbb{N}$ denote the set of positive integers. Consider functions p and q from $\mathbb{N}$ to itself such that $p(1) = q(3) = 2$, $p(2) = q(1) = 3$, $p(3) = q(2) = 4$, $p(4) = q(4) = 1$, and $p(n) = q(n) = n$ for $n \geq 5$.

   (a) Find a function $f : \mathbb{N} \to \mathbb{N}$ such that $f(f(n)) = p(n) + 2$.

   (b) Determine if there is a function $g : \mathbb{N} \to \mathbb{N}$ such that $g(g(n)) = q(n) + 2$.