

# Lectures on Challenging Mathematics

## Introduction to Math Olympiads

### Number Theory

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## 1.4 Mathematical reasoning and number theory

1. Compute  $\gcd(1, 91) + \gcd(2, 91) + \cdots + \gcd(91, 91)$ . In general, let  $p$  and  $q$  be two distinct primes, express  $\gcd(1, pq) + \gcd(2, pq) + \cdots + \gcd(pq, pq)$  in closed form.
2. More than 2000 year ago, Euclid first introduced the fact of the existence of infinitely many primes. Ryan vaguely recalls a proof he has seen before:

For the sake of the contradiction, assume that there are only finitely many primes  $p_1, p_2, \dots, p_n$ . Then  $p_1 p_2 \dots p_n + 1$  (or  $p_1 p_2 \dots p_n - 1$ ) must also be a prime . . .

Alex does not agree with Ryan's proof, and challenges the fact that  $p_1 p_2 \dots p_n + 1$  (or  $p_1 p_2 \dots p_n - 1$ ) being a prime. But Ryan claims that it can be established by induction. Can you help Alex and Ryan to sort out this proof?

3. Let  $p$  and  $q$  be two consecutive odd prime numbers. Prove that  $p + q$  is a product of at least three (not necessarily distinct) primes.
4. Numbers from 1 to 81 are placed in the squares of a  $9 \times 9$  table. Ben found products of all numbers in each row, while Cindy found products of all numbers in each column. Could their lists of products coincide?
5. Let  $a, b, c, d$  be positive integers such that  $ab = cd$ . Prove that  $a + b + c + d$  cannot be a prime.