Lectures on Challenging Mathematics

Introduction to Math Olympiads

Number Theory

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1.4 Mathematical reasoning and number theory

1. Compute $\gcd(1,91)+\gcd(2,91)+\cdots+\gcd(91,91)$. In general, let p and q be two distinct primes, express $\gcd(1,pq)+\gcd(2,pq)+\cdots+\gcd(pq,pq)$ in closed form.

2. More than 2000 year ago, Euclid first introduced the fact of the existence of infinitely many primes. Ryan vaguely recalls a proof he has seen before:

   > For the sake of the contradiction, assume that there are only finitely many primes $p_1, p_2, \ldots, p_n$. Then $p_1p_2\ldots p_n + 1$ (or $p_1p_2\ldots p_n - 1$) must also be a prime ...

   Alex does not agree with Ryan's proof, and challenges the fact that $p_1p_2\ldots p_n + 1$ (or $p_1p_2\ldots p_n - 1$) being a prime. But Ryan claims that it can be established by induction. Can you help Alex and Ryan to sort out this proof?

3. Let p and q be two consecutive odd prime numbers. Prove that $p+q$ is a product of at least three (not necessarily distinct) primes.

4. Numbers from 1 to 81 are placed in the squares of a 9×9 table. Ben found products of all numbers in each row, while Cindy found products of all numbers in each column. Could their lists of products coincide?

5. Let a, b, c, d be positive integers such that $ab = cd$. Prove that $a+b+c+d$ cannot be a prime.