

Lectures on Challenging Mathematics

Elements of Math Olympiads

Number Theory

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“Cogito ergo Sum” – “I think, therefore I am”

René Descartes (1596–1650)

“Success is not final, failure is not fatal, it is the courage to continue that counts.”

Winston Churchill (1874–1965)

“I can see that without being excited, mathematics can look pointless and cold. The beauty of mathematics only shows itself to more patient followers.”

Maryam Mirzakhani (1977–2017)

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1.11 Diophantine equations (part 1)

- Sarah intended to multiply a two-digit number and a three-digit number, but she left out the multiplication sign and simply placed the two-digit number to the left of the three-digit number, thereby forming a five-digit number. This number is exactly nine times the product Sarah should have obtained. Find the five-digit number that Sarah got.
- Private Ryan is working on the following problem:

What is the largest positive integer n for which there is a unique integer k such that

$$\frac{8}{15} < \frac{n}{n+k} < \frac{7}{13}?$$

- (a) Ryan rewrote the given inequalities as

$$\frac{208}{390} < \frac{n}{n+k} < \frac{210}{390}$$

and claimed that $n = 209$ is the numerical answer to the problem. Unfortunately, $n = 209$ is not correct. Complete the statement by Captain Miller to spot the subtle flaw in Ryan's reasoning: Knowing that n is _____ does not necessarily imply that k is _____.

- (b) Ryan was able to modify the given inequalities to obtain the correct numerical answer of $n = 112$ using a very similar approach as in (a). How did he do it?
- (c) The above approaches are not well-reasoned/written math solutions. Complete the following sentences for a complete solution.

We can see that the given inequalities are equivalent to

$$48n < \underline{\hspace{2cm}} k < 49n.$$

Consequently, the problem is to find the _____ open interval $(48n, 49n)$ that contains _____ one integral multiple of 56.

Because the _____ of the interval is _____, it contains _____ integers. If $n - 1 \geq \underline{\hspace{2cm}}$, the interval will contain at least _____ multiples of 56. Hence _____ is the _____ candidate for n . Indeed, we find that

$$48 \cdot 112 = 56 \cdot 96 < 56 \cdot 97 < 56 \cdot 98 = 49 \cdot 112,$$

also exhibiting that _____ is the unique positive integer corresponding to $n = 112$.

- Let \mathcal{R} denote the region enclosed by the graphs of $y = x$ and $2^{100}y = x^2$. Determine the number of points of the form $(2^m, 2^n)$, where m and n are positive integers, inside \mathcal{R} (do not include the boundary of \mathcal{R}).

4. In the following 3×3 array of distinct positive integers, the products of entries in each row, each column, and each diagonal are all equal to n . Find the least possible value of n .

$$\begin{array}{ccc} a & b & c \\ d & e & f \\ g & h & i \end{array}$$

5. Let x and y be positive integers with $x < y$. Find all possible integer values of

$$P = \frac{x^3 - y}{1 + xy}.$$

1.13 Bézout's Identity and Frobenius Coin Theorem (part 1)

1. For two given positive relatively prime integers a and b , we call an integer n *representable* by a and b if there are nonnegative integers x and y such that $n = ax + by$.
 - (a) When asked to determine if 26 is the greatest integer that is not representable by 7 and 9, Luzor announced a plan: "It is enough to check if each of the next seven numbers is representable by 7 and 9." Do you follow Luzor's logic?
 - (b) What is the greatest integer that is not representable by 7 and 9?
2. (Continuation) Determine if each of the following statements is true:
 - (a) If integers m and n are both representable by a and b , then their sum $m + n$ is always representable by a and b .
 - (b) If integers m and n are both not representable by a and b , then their sum $m + n$ is always not representable by a and b .
 - (c) If positive integer m is not representable by a and b and $m = ax + by$ for some integers a and b , then $xy \leq 0$.
 - (d) If integer m is representable by a and b and $m = ax + by$ for some integers a and b , then $xy \geq 0$.
 - (e) Consider a block of a consecutive integers starting at m . If every integer in the block is representable by a and b , then all integers greater than or equal to m are representable by a and b .
3. (Continuation) Determine which positive integers are and aren't representable by 5 and 7. Repeat this exercise for 4 and 9. Fill in all the entries in the following table.

nonnegative integers	5 and 7	4 and 9
representable by		
not representable by		

Make a conjecture about the greatest integer that cannot be represented by a pair of relatively prime numbers a and b . Also, make a conjecture about the number of nonnegative integers that cannot be represented using such a and b .

4. Let a and b be two relatively prime positive integer. What can you say about the remainders when the numbers $a, 2a, \dots, (b-1)a$ are divided by b ?
5. Let a and b are two relatively prime positive integers.
 - (a) Explain that we can always find a pair of integer solutions (x_0, y_0) such that $ax_0 + by_0 = 1$.
 - (b) Describe all integer solutions to the equation $ax + by = 1$ in terms of (x_0, y_0) .