Lectures on Challenging Mathematics

Elements of Math Olympiads

Geometry

Summer 2021

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"Cogito ergo Sum" – *"I think, therefore I am"*

René Descartes (1596–1650)

"Success is not final, failure is not fatal, it is the courage to continue that counts."

Winston Churchill (1874–1965)

"I can see that without being excited, mathematics can look pointless and cold. The beauty of mathematics only shows itself to more patient followers."

Maryam Mirzakhani (1977–2017)

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1.6   Angle chasing and the centers of triangles (part 1)

1. Let ABC be an acute triangle, and let AD, BE, CF be the three altitudes of the triangle. Denote by H the orthocenter of the triangle. Determine the number of circles that passes through four of the seven labeled points in this diagram (namely, A, B, C, D, E, F, H).

2. Let ABC be a triangle with circumcircle ω. Let O, I, I_A be its circumcenter, incenter, and excenter opposite to vertex A, respectively. Let M be the midpoint of $\widehat{BC}$ (not including A), and let N be the midpoint of side BC.

   (a) Explain why points O, M, N are collinear, and why points A, I, M, I_A are collinear.

   (b) Explain the following fact: In a triangle, the interior angle bisector at one of its vertices lies *in between* the median and the altitude from the vertex.

   (c) Show that points B, C, I, I_A lie on a circle and determine the center of this circle.

3. (Continuation) The angle-bisector AI intersects BC at D. Prove that $MD \cdot MA = MI^2$.

4. In triangle ABC, points L, M, N are the midpoints of sides BC, CA, AB, respectively. Show that $\angle LAC = \angle ABM$ if and only if $\angle ANC = \angle ALB$.

5. Circles ω_1 and ω_2 intersect at points P and Q. Line ℓ_1 is tangent to ω_1 and ω_2 at A and B, respectively; line ℓ_2 is tangent to ω_1 and ω_2 at C and D respectively. Line ℓ passes through the centers of the two circles, and intersects segments AC and BD at M and N, respectively. Show that $PMQN$ is a rhombus.
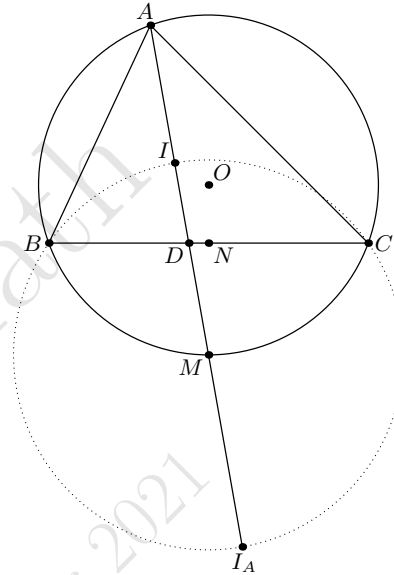
1.14   Brahmagupta's formula and geometric computations

1. Quadrilateral $ABCD$ is inscribed in a circle. Given that $AB = a$, $BC = b$, $CD = c$, and $DA = d$. Write $\cos C$ and $\sin C$ in terms of a, b, c, d, respectively. Establish *Brahmagupta's formula*:

$$
\begin{aligned}
[ABCD] &= \sqrt{(s-a)(s-b)(s-c)(s-d)} \\
&= \frac{\sqrt{(a+b+c-d))(b+c+d-a)(c+d+a-b)(b+c+d-a)}}{4},
\end{aligned}
$$

   where s denotes the semi-perimeter $(a+b+c+d)/2$.

2. (Continuation) Explain why Heron's formula is a special case of Brahmagupta's formula. Brahmagupta's formula reveals an interesting fact: The order of the side lengths does not effect the area of a cyclic quadrilateral. How do we establish this fact without knowing Brahmagupta's formula? Geometric constructions leading to this fact could be useful in a more natural proof of Ptolemy's theorem, arguably the most famous theorem for cyclic quadrilateral.

3. (Continuation) In a similar way, we can establish *Bretschneider's formula*, which generalizes Brahmagupta's formula, states that the area of an arbitrary quadrilateral with side lengths a, b, c, and d, is given by

$$
\sqrt{(s-a)(s-b)(s-c)(s-d) - abcd(\cos\theta)^2},
$$

   where $s = (a+b+c+d)/2$ and θ is half the sum of either pair of opposite angles. At this point, we will accept this result without formal proof. What fact can be deduced from Bretschneider's formula? (Bretschneider's formula was developed by German mathematician Carl A. Bretschneider in 1842.)

   Hint: This fact can help us quickly solve one of the following problems.

   Query: How to prove Bretschneider's formula?

4. In rectangle $ABCD$, $AB = 2$ and $AD < 2$. Denote by ω the circle centered at A with radius 2. Circle ω intersects ray AD and segment CD at E and F respectively. Let $\mathcal{R}_1$ denote the (convex) region bounded by segments DE, DF and minor arc $\overset{\frown}{EF}$, and let $\mathcal{R}_2$ denote the (concave) region bounded by segments CB, CF and minor arc $\overset{\frown}{BF}$. Given that $\mathcal{R}_1$ can be cut out and moved to exactly fit in $\mathcal{R}_2$, with D matching C and $\overset{\frown}{EF}$ being tangent to $\overset{\frown}{BF}$. Find the length of AD.
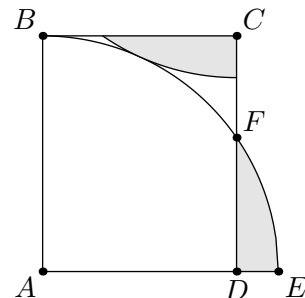
5. Consider all quadrilaterals $ABCD$ such that $AB = 14$, $BC = 9$, $CD = 7$, and $DA = 12$. What is the radius of the largest possible circle that fits inside or on the boundary of such a quadrilateral?