

Lectures on Challenging Mathematics

Elements of Math Olympiads

Combinatorics

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“Cogito ergo Sum” – “I think, therefore I am”

René Descartes (1596–1650)

“Success is not final, failure is not fatal, it is the courage to continue that counts.”

Winston Churchill (1874–1965)

“I can see that without being excited, mathematics can look pointless and cold. The beauty of mathematics only shows itself to more patient followers.”

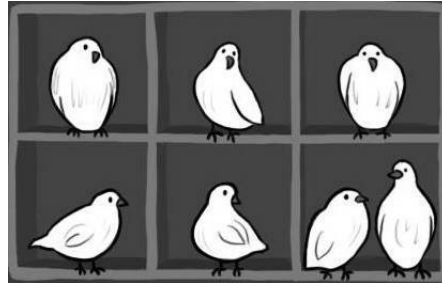
Maryam Mirzakhani (1977–2017)

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1.4 The first tour of the Pigeonhole principle

- To stuff seven pigeons into six holes requires that at least two pigeons be stuffed into the same hole, because, on average, there are $\frac{7}{6}$ pigeons in each hole. Since $\frac{7}{6} > 1$, some holes must have at least two pigeons.



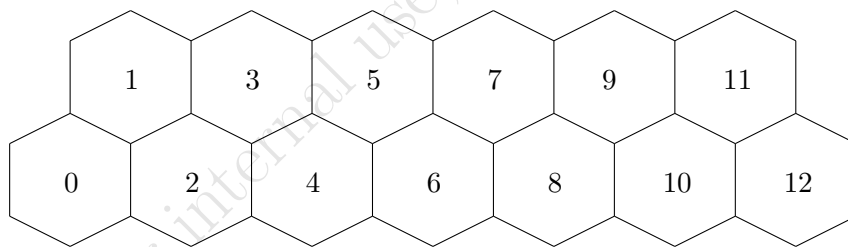
The *pigeonhole principle* states that to stuff $n + 1$ pigeons into n holes requires that at least two pigeons to be stuffed into the same hole. More generally, given positive integers m and n with $m > n$, to stuff m pigeons into n hole requires some hole contains at least $\lceil m/n \rceil$ (the least integer greater than or equal to m/n) pigeons. Furthermore, some hole contains at most $\lfloor m/n \rfloor$ (the greatest integer less than or equal to m/n) pigeons. Simple and intuitive, it can be used to demonstrate possibly unexpected results.

When rolling several dice the score is the sum of them. How many times must we roll two dice in order to be sure to get the same score two times?

- Show that any set of
 - 51 different positive integers, all less than or equal to 100, contains a pair of consecutive numbers.
 - 27 different positive odd integers, all less than 100, contains a pair whose sum is 102.
 - 20 different integers chosen from the arithmetic progression $1, 4, 7, \dots, 100$ contain a pair whose sum is 104.
- The squares of an 8×8 checkerboard are filled with the numbers $\{1, 2, \dots, 64\}$. Prove that some two adjacent squares (sharing a side) contain numbers differing by at least 5.
- Set S contains five integers each greater than 1 and less than 120. Show that S contains a prime or two elements of S share a prime divisor.
- Let n be a positive integer, and let S be a subset of $\{1, 2, 3, \dots, 2n\}$ containing $n + 1$ elements.
 - Show that there are two distinct relatively prime elements of S . (Want a hint? Well, a hint is already given in a prior problem.)
 - Show that there are two distinct elements of S such that one divides the other.

1.18 Recursive counting (part 2)

- A bowl contains a mixture of r red and b brown candies. Find values of r and b so that there is exactly a 50 percent chance that the colors of two randomly chosen candies will not match. There are many possibilities, including $r = 15$ and $b = 10$. Verify that these values work, then find other values for r and b (besides $r = 10$ and $b = 15$) that are consistent with this information. You could try small values for r and b .
- Lazy Linus doesn't like doing homework. However, he has to do homework for at least two days for every consecutive three-day block in order not to fail his class. In how many ways can he choose some days (possibly none) out of 14 consecutive days to slack off and not do homework, and still pass his class? Solve this problem by
 - considering the number of days Linus can slack off;
 - using a recursion.
- Consider the honeycomb pattern shown below, which consists of two rows of numbered hexagons. A honeybee crawls from hexagon number 0 to hexagon number 12, always moving from one hexagon to an adjacent hexagon whose number is greater. We want to find the number of different paths for the honeybee to achieve his goal.
 - Let $p(n)$ denote the number of paths from hexagon 0 to hexagon n , and assume that $p(0) = 1$. It is easy to see the values of $p(1)$, $p(2)$, and $p(3)$. You should also see a pattern emerging. Establish this pattern.
 - Express $p(n)$ as the sum of binomial coefficients. Note: It might be useful to represent each path from hexagon 0 to hexagon n with k moves to the right, ℓ moves up, and m moves down.



- Let $\{F_n\}_{n=1}^{\infty}$ be the Fibonacci sequence defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$. Fibonacci numbers can be obtained as the sums of the binomial coefficients. Construct a Pascal triangle, visualize and prove this result:

$$F_{n+1} = \sum_{k=0}^n \binom{n-k}{k}.$$

5. A collection of ten cubes consists of one cube with edge-length k for each integer k , with $1 \leq k \leq 10$. A tower is to be built using all ten cubes according to the rules:
- Any cube may be the bottom cube of the tower.
 - The cube immediately on top of a cube with edge-length k must have edge-length at most $k + 2$.

Find the number of different towers that can be constructed.