

Lectures on Challenging Mathematics

Elements of Math Olympiads

Algebra

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“Cogito ergo Sum” – “I think, therefore I am”

René Descartes (1596–1650)

“Success is not final, failure is not fatal, it is the courage to continue that counts.”

Winston Churchill (1874–1965)

“I can see that without being excited, mathematics can look pointless and cold. The beauty of mathematics only shows itself to more patient followers.”

Maryam Mirzakhani (1977–2017)

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1.7 Introduction to functional properties (part 2)

1. Match functions with functional identities they satisfy.

(a) $f(x) = x^3 + x$

(i) $f(x)f(a) = f(x + a)$

(b) $f(x) = \pi^x$

(ii) $f\left(\frac{x}{a}\right) = f(x) - f(a)$

(c) $f(x) = 2^x + 2^{-x}$

(iii) $f(-x) = f(x)$

(d) $f(x) = \log x$

(iv) $f(-x) = -f(x)$

2. Some functions f have the property that $f(-x) = f(x)$ for all values of x . Such a function is called *even*. Give a few different types of examples – some of your examples should reflect the choice of the term *even*. Some functions f have the property that $f(-x) = -f(x)$ for all values of x . Such a function is called *odd*. Give a few different types of examples – some of your examples should reflect the choice of the term *odd*.

Each of the functional relations (or identities or equations) $f(-x) = f(x)$ and $f(-x) = -f(x)$ is a concise way to describe a certain symmetry property of a graph $y = f(x)$. In particular, one describes a mirror reflection, and the other describes a *centrally symmetric* property (or a *half-turn symmetry*); that is, the graph coincides with itself by applying a 180° rotation at a certain point. Do you know which one is which and why?

3. Given that function f satisfies the identity $f(50 + x) = f(50 - x)$ for all x -values, show that the graph $y = f(x)$ has reflective symmetry. Identify the axis of reflection. Name two such functions. This functional identity is equivalent to which of the following functional identities?

(a) $f(x) = f(100 - x)$

(b) $f(x + 20) = f(120 - x)$

(c) $f(x + 20) = f(80 - x)$

(d) $f(x + 20) = f(x - 120)$

4. Function $f : \mathbb{R} \rightarrow \mathbb{R}$ is odd and its graph is symmetric about the line $x = 36$. We can show that f is periodic by the following steps:

(a) Write a functional equation indicating that f is odd. Write a functional equation indicating that the graph of f has a symmetry at $x = 36$.

(b) Show that $f(72 + x) = -f(x)$ and then show that $f(x)$ is periodic.

5. The following problem appeared in 2016 AMC10A.

A binary operation \diamond has the properties that $a \diamond (b \diamond c) = (a \diamond b) \cdot c$ and that $a \diamond a = 1$ for all nonzero real numbers a, b , and c . (Here \cdot represents multiplication). Find x such that $2016 \diamond (6 \diamond x) = 100$.

(a) Express the operation \diamond in the language of functional equations. You may start with the following: Let $f(x, y)$ be a function defined on $\mathbb{R}^* \times \mathbb{R}^*$, where \mathbb{R}^* denotes the set of nonzero numbers, such that

(b) Determine the \diamond operation; that is, determine the function $f(x, y)$.

1.15 Recursive relations and inductive process

1. Let a_1, a_2, \dots be a sequence defined by $a_1 = 1$, $a_2 = 2$, and $a_{n+2} = 2a_{n+1} + 3a_n$ for $n \geq 1$. Consider the *partial* sums

$$s_1 = \frac{a_1}{7}, \quad s_2 = \frac{a_1}{7} + \frac{a_2}{7^2}, \quad \dots, \quad s_k = \sum_{n=1}^k \frac{a_n}{7^n} = \frac{a_1}{7} + \frac{a_2}{7^2} + \dots + \frac{a_k}{7^k}$$

for every positive integer k .

- (a) Clearly, (s_1, s_2, \dots) is an increasing sequence. Show that the sequence is also *bounded*; that is, there is a constant c such that $s_k < c$ for every positive integer k .
- (b) Because (s_1, s_2, \dots) is increasing and bounded, conceptually, we can see that this sequence *approaches* a finite *limiting* value. This limiting value is the *limit* of this sequence and it is denoted by

$$S = \sum_{n=1}^{\infty} \frac{a_n}{7^{n+1}}.$$

Find a way to convince your peers that $S = \frac{7}{32}$.

2. Prove that any given integer n , there are infinitely many integers m such that $n = \pm 1^2 \pm 2^2 \pm 3^2 \pm \dots \pm m^2$ for a suitable choice of the $+$ and $-$ signs.
3. An ant stands on vertex A of a regular tetrahedron $ABCD$. At every step, the ant moves from its current vertex to one of the three adjacent vertices, each move occurring with equal probability. Find the probability that the bug is on vertex A after 100 steps.
4. Sixty-four balls are separated into several piles. Each step we are allowed to apply the following operation. Pick two piles, say pile \mathcal{A} with p balls and pile \mathcal{B} with q balls and $p \geq q$, and then remove q balls from pile \mathcal{A} and put them in pile \mathcal{B} . Prove that it is possible to put all the balls into one pile.
5. Shanille O'Keal shoots free-throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability she hits exactly 50 of her first 100 shots?