Lectures on Challenging Mathematics

Introduction to Math Olympiads

Algebra

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1.5 The first look at the AM-GM inequality

1. Let $a$ and $b$ be positive real numbers. Show that $a + b \geq 2\sqrt{ab}$. Sketch the graph of

$$f(x) = \frac{7x^2 + 4}{x}$$

by using an asymptotic line and an asymptotic hyperbola. Determine the extreme values of $f(x)$.

2. Let $a, b, c, d$ be positive real numbers. Prove that

$$\frac{a + b + c + d}{4} \geq \sqrt[4]{abcd} \geq \frac{1}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}}$$

3. Let $n$ be a positive integer. For positive integers $a_1, a_2, \ldots, a_n$, define their arithmetic mean as

$$A_n = \frac{a_1 + a_2 + \cdots + a_n}{n},$$

their geometric mean as

$$G_n = \sqrt[n]{a_1 a_2 \cdots a_n},$$

and their harmonic mean as

$$H_n = \frac{1}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}.$$  

The AM-GM inequality states that

$$A_n \geq G_n.$$  

We will establish this inequality when we introduce mathematical induction. Assume that the AM-GM inequality is true, prove the GM-HM Inequality:

$$G_n \geq H_n.$$  

4. Let $a, b, c$ be real numbers with $a \geq b > 1$ and $0 < c < \pi$. Determine the respective extreme values of

$$\log_a \left( \frac{a}{b} \right) + \log_b \left( \frac{b}{a} \right) \quad \text{and} \quad \frac{9c^2 \sin^2 c + 4}{c \sin c}.$$  

5. Let $x, y, z > 1$ be real numbers with $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$. Prove that $(x - 1)(y - 1)(z - 1) \geq 8$.  