Lectures on Challenging Mathematics

Introduction to Math Olympiads

Algebra

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1.5 The first look at the AM-GM inequality

1. Let a and b be positive real numbers. Show that $a+b \geq 2\sqrt{ab}$. Sketch the graph of

$$f(x) = \frac{7x^2+4}{x}$$

by using an asymptotic line and an asymptotic hyperbola. Determine the extreme values of $f(x)$.

2. Let a, b, c, d be positive real numbers. Prove that

$$\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd} \geq \frac{1}{\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}}{4}}.$$

3. Let n be a positive integer. For positive integers $a_1, a_2, \ldots, a_n$, define their *arithmetic mean* as

$$A_n = \frac{a_1+a_2+\cdots+a_n}{n},$$

their *geometric mean* as

$$G_n = \sqrt[n]{a_1 a_2 \cdots a_n},$$

and their *harmonic mean* as

$$H_n = \frac{1}{\frac{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}{n}}.$$

The *AM-GM inequality* states that

$$A_n \geq G_n.$$

We will establish this inequality when we introduce mathematical induction.

Assume that the AM-GM inequality is true, prove the *GM-HM Inequality*:

$$G_n \geq H_n.$$

4. Let a, b, c be real numbers with $a \geq b > 1$ and $0 < c < \pi$. Determine the respective extreme values of

$$\log_a\left(\frac{a}{b}\right) + \log_b\left(\frac{b}{a}\right) \quad \text{and} \quad \frac{9c^2\sin^2 c+4}{c\sin c}.$$

5. Let $x, y, z > 1$ be real numbers with $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$. Prove that $(x-1)(y-1)(z-1) \geq 8$.