Lectures on Challenging Mathematics

Calculus (part 1)

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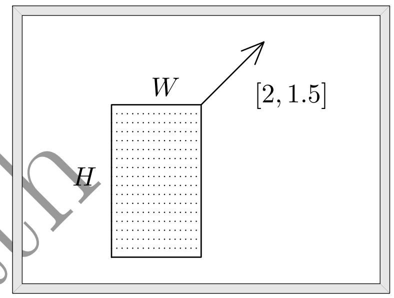
Contents

3.16 Operational rules for derivatives (part 8)

1. Without using a calculator, find derivatives of the following:

   (a) $f(t) = \ln\sqrt{t}$ (b) $g(x) = \left(\sqrt{2x}\right)^3 + \sqrt{x^3}$ (c) $h(u) = u^2\sqrt{u}$

2. Kelly is using a mouse to enlarge a rectangular frame on a computer screen. As shown at right, Kelly is dragging the upper right corner at 2 cm per second horizontally and 1.5 cm per second vertically. Because the width and height of the rectangle are increasing, the enclosed area is also increasing. At a certain instant, the rectangle is 11 cm wide and 17 cm tall. By how much does the area increase during the next 0.1 second? Make calculations to show that most of the additional area comes from two sources — a contribution due solely to increased width, and a contribution due solely to increased height. Your calculations should also show that the rest of the increase is insignificant — amounting to less than 1%.



3. (Continuation) Repeat the calculations, using a time increment of 0.001 second. As above, part of the increase in area is due solely to increased width, and part is due solely to increased height. What fractional part of the change is not due solely to either effect?

4. (Continuation) Let $A(t) = W(t) \cdot H(t)$, where A, W, and H stand for area, width, and height, respectively. The previous examples illustrate the validity of the equation

$$\Delta A = W \cdot \Delta H + H \cdot \Delta W + \Delta W \cdot \Delta H,$$

   in which the term $\Delta W \cdot \Delta H$ plays an insignificant role as $\Delta t \to 0$. Divide both sides of this equation by Δt and find limits as $\Delta t \to 0$, thus showing that the functions $\frac{dA}{dt}$, W, H, $\frac{dW}{dt}$, and $\frac{dH}{dt}$ are related in a special way. This relationship illustrates a theorem called the *Product Rule*.

5. For each pair of functions $f(x)$ and $g(x)$ below, find the derivative of the product $f(x)g(x)$ in two way: one with the Product rule and one by mental math.

   (a) $(f(x), g(x)) = (2, \sin x)$ (b) $(f(x), g(x)) = (x^3, x^5)$

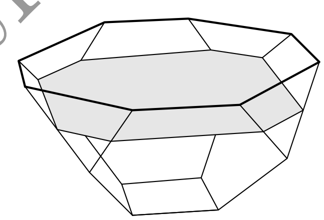
   (c) $(f(x), g(x)) = (x, \sqrt{x})$ (d) $(f(x), g(x)) = (\cos x, \tan x)$

4.19 Introduction to differential equations (part 10)

1. Express the surface area S of a sphere as a function of its volume V, then find $\dfrac{dS}{dV}$.

2. Consider the well-known formula $V = \pi r^2 h$ for the volume of a cylinder. This expresses V as a function of r and h. Calculate $\dfrac{dV}{dr}$ and $\dfrac{dV}{dh}$. What assumptions did you make? Interpret each derivative geometrically by making appropriate diagrams. In particular, explain the geometrical content of the approximation $\Delta V \approx \dfrac{dV}{dr} \cdot \Delta r$ and the equation $\Delta V = \dfrac{dV}{dh} \cdot \Delta h$. Explain why one equation is exact and the other is only an approximation.

3. The bowl shown in the figure is 3 feet deep, and is situated so that its octagonal opening is parallel to the ground. Its volume is 125 cubic feet. Left alone, the water will evaporate at a rate that is *proportional to the area of the water surface*.



   (a) Explain why this should be expected.

   (b) Suppose that $\dfrac{dV}{dt} = -0.12A$, where $V(t)$ and $A(t)$ are the volume and surface area after t days of evaporation. In what units should the constant -0.12 be expressed?

   (c) By considering what happens during short intervals of time, deduce that the water depth $y(t)$ must decrease at a *constant* rate.

   (d) Given $y(0) = 3$, calculate how many days pass until evaporation empties the bowl.

4. A particle moves along a number line according to $x = t^4 - 4t^3 + 3$, during the time interval $-1 \le t \le 4$. Calculate the velocity function $\dfrac{dx}{dt}$ and the acceleration function $\dfrac{d^2x}{dt^2}$. Use them to help you give a detailed description of the position of the particle:

   (a) At what times is the particle (instantaneously) at rest, and where does this happen?

   (b) During what time intervals is the position x increasing? When is x decreasing?

   (c) At what times is the acceleration of the particle zero? What does this signify?

   (d) What is the complete range of positions of the particle?

   (e) What is the complete range of velocities of the particle?

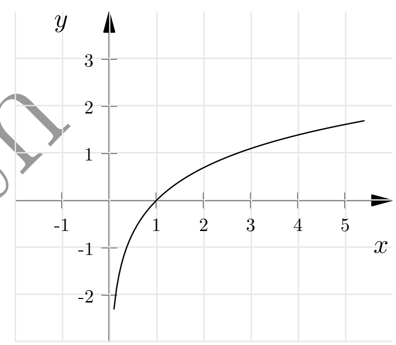
5. Find $\dfrac{dy}{dx}$ for the function defined implicitly by $y = xy^2 + 1$.

5.2 Properties of derivatives (part 11)

1. By comparing the derivatives of $\sin x$ and $\sin(2x)$, e^x and e^{2x}, $x^{\frac{3}{5}}$ and $2x^{\frac{3}{5}}$, Private Ryan claims that he can find the antiderivatives of $\sin 2x$, $3e^{5x}$, and $7x^9$. Can you do that too?

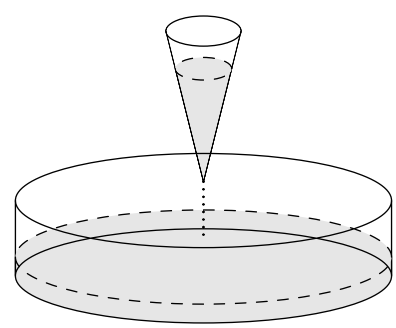
   Private Ryan also thinks the antiderivative of $\cos(x^2)$ is $\dfrac{\sin(x^2)}{2x}$. Is he correct? What is the rationale behind his thinking?

2. If the graph of $f(x) = \ln x$, denoted by $\mathcal{C}$, shows the elevation during a hike along a mountain ridge, as a function of time. The hike is getting *easier* all the time – because it gets less steeper along the way. Mathematically speaking, $\mathcal{C}$ never changes its *concavity*. Indeed, it slope function $f(x)' = \frac{1}{x}$ keep decreasing and hence it is always *concave up*. (As we mentioned before, the graph bends like a bowl that "pills water".) What is the best way to describe the fact that the derivative of a function f keeps increasing?



3. (Continuation) Let P be a point on $\mathcal{C}$ and let ℓ_P denote the tangent line of $\mathcal{C}$ at P. A remarkable fact is that $\mathcal{C}$ lies completely under ℓ_P. (We shall establish this fact when we study inflection point and concavity formally.)
   (a) Use that fact that $f'' < 0$ to convince yourself why does this fact make sense.
   (b) Determine the point $P = (x, f(x))$ on $\mathcal{C}$ where the ratio $\dfrac{f(x)}{x}$ reaches its largest value.
   (c) Conclude that $x^{\frac{1}{x}} \le e^{\frac{1}{e}}$ for $x > 0$.

4. A cylindrical tank that has a base area 400π square feet is placed on a level surface. A right cone shaped conical tank with height 12 feet and diameter 8 feet is placed over the cylindrical tank. The tip of the cone is pointing downward perpendicular towards to the level surface. Water is draining from the tip of a conical tank into the cylindrical tank. At the moment that the depth of the water in the conical tank is h feet, the depth of the water is changing at the rate of $(h - 12)$ feet per minute.
   (a) Write an expression for the volume of water in the conical tank as a function of h.
   (b) At what rate is the volume of water in the conical tank changing when $h = 3$? Indicate units of measure.
   (c) Let y be the depth, in feet, of the water in the cylindrical tank. At what rate is y changing when $h = 3$? Indicate units of measure.



5. Show that if the function $y = f(x)$ is differentiable at $x = a$ and $f(a)$ is a *local extreme* (that is, it is either a *local minimum* or a *local maximum*), then $f'(a) = 0$.