Lectures on Challenging Mathematics

Calculus (part 1)

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Contents

1.5 Algebraic knowledge and tools (part 3)

1. Given that the x-intercepts of the graph of $y = f(x)$ are -3, 2 and 7, find an example of $f(x)$.

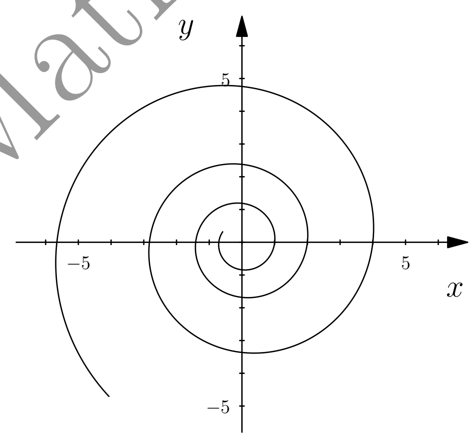
2. If $f(x)$ is an odd function and $g(x)$ is an even function, what can be said about the following functions.

   (a) $f(f(x))$ (b) $g(g(x))$ (c) $f(g(x))$ (d) $g(f(x))$

3. Working in radian mode, determine all θ, with $0 \le \theta < 2\pi$, such that

   (a) $\sin(2\theta) < \sin\theta$ (b) $\sin(2\theta) = 2\sin\theta$ (c) $\sin(2\theta) \ge \sin\theta$

4. It is awkward to describe fundamental curves like *spirals* using only the Cartesian coordinates x and y. The example shown, on the other hand, is easily described with polar coordinates – all its points fit the equation $r = 2^{\theta/360}$ (using degree mode). Choose three specific points in the diagram and make calculations that confirm this. What range of θ-values does the graph represent? Use a graphing tool to obtain pictures of this spiral.



5. A function f defined on a subset of the real numbers with real values is *(monotonically) nonincreasing* or *(monotonically) decreasing* on interval $[a, b]$ if $f(x) \ge f(y)$ for $x \le y$. We say function is *strictly* decreasing on $[a, b]$ if $f(x) > f(y)$ for $x < y$. Likewise, we can define *nondecreasing* (or *increasing* and *strictly increasing* functions. We call a function *monotonic* if it is either entirely nonincreasing or nondecreasing on its domain. We call a function *strictly monotonic* if it is either entirely increasing or decreasing on its domain.

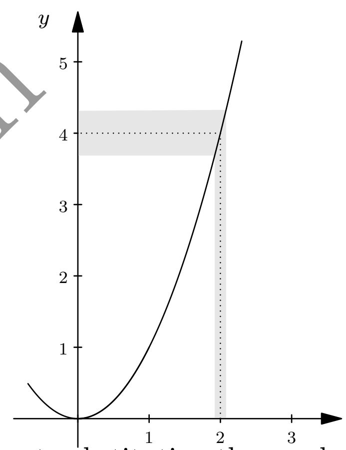
   Find a function with domain $\mathbb{R}$ (the set of real numbers) which is

   (a) a strictly increasing exponential function;
   (b) a strictly decreasing exponential function;
   (c) a strictly increasing logarithm function;
   (d) a strictly decreasing logarithm function;
   (e) a strictly increasing power function;
   (f) a strictly decreasing power function;
   (g) both decreasing and increasing (that is, both nonincreasing and nondecreasing);

1.18 Limits and approximations (part 9)

1. *One-sided limits.* Consider the *sign function*, which is defined for all nonzero values of x by $\operatorname{sgn}(x) = \dfrac{x}{|x|}$. Confirm that both $\lim\limits_{x\to 0^-} \operatorname{sgn}(x)$ and $\lim\limits_{x\to 0^+} \operatorname{sgn}(x)$ exist, and notice that they have different values. Because the two one-sided limits do not agree, $\operatorname{sgn}(x)$ does not approach a (two-sided) limit as $x \to 0$. (Notice that $\operatorname{sgn}(0)$ remains undefined, hence it is *discontinuous* at $x = 0$.)

2. Verify that $|x^3 - 8| < 0.0001$ is true for all numbers x that are sufficiently close to 2. In other words, show that the inequality is satisfied if the distance from 2 to x is small enough. How small is "small enough"?
What does "limit" mean? Let p be any small positive number. Show that there is another small positive number d, which depends on p, that has the following property: whenever $|x - 2| < d$ it is true, it is also true that $|x^3 - 8| < p$. This means —intuitively — that x^3 can be brought arbitrarily close to 8 by making x suitably close to 2. It is customary to summarize this situation by writing $\lim\limits_{x\to 2} x^3 = 8$.



3. One might think that the concept of limit is simple: It is just about substituting the x value. For example, it is routine to find the following limits. Do so.

   (a) $\lim\limits_{a\to 2} \dfrac{2a^2 - 3a - 2}{2a + 1}$ (b) $\lim\limits_{b\to 3} \dfrac{\sqrt{b^3 - 8}}{\sqrt{b^2 - 4}}$

   (c) $\lim\limits_{c\to \frac{\pi}{2}} \dfrac{2c}{\sin c}$ (d) $\lim\limits_{d\to \pi} \dfrac{d}{\sin \frac{d}{2}}$

4. (Continuation) The above limits are easy to find only because we are working with functions that are *continuous* at the points of interest. Compute the following limits instead. These limits are all of *indeterminate form*. (Note that we emphasize the term "*form*" rather than the term "*value*", because we can determine the values of these limits.)

   (a) $\lim\limits_{a\to -0.5} \dfrac{2a^2 - 3a - 2}{2a + 1}$ (b) $\lim\limits_{b\to 2} \dfrac{\sqrt{b^3 - 8}}{\sqrt{b^2 - 4}}$ (c) $\lim\limits_{c\to 0} \dfrac{2c}{\sin c}$

   (d) $\lim\limits_{d\to 0} \dfrac{d}{\sin \frac{d}{2}}$ (e) $\lim\limits_{e\to -0.5} \dfrac{1 + e}{(2e + 1)^2}$ (f) $\lim\limits_{f\to -0.5} \dfrac{1 + f}{2f + 1}$

5. Consider two positive sequences $(a_1, a_2, \dots)$ and $(b_1, b_2, \dots)$. Explain why statements like "a_n is very close to b_n" or "a_n and b_n are approximately the same" are very vague.

   Compare the following expressions. How do they differ from each other?

   (a) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n$ (b) $\lim\limits_{n\to\infty} (a_n - b_n) = 0$ (c) $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 1$

2.20 Limits, slopes, and derivatives (part 10)

1. Riding a train that is traveling at 72 mph, Morgan walks at 4 mph toward the front of the train, in search of the snack bar. How fast is Morgan traveling, *relative to the ground*?

2. (Continuation) Given *differentiable* functions f and g, let $k(t) = f(t)+g(t)$. Use the definition of the derivative to show that $k'(t) = f'(t) + g'(t)$ must hold. This justifies *term-by-term differentiation*.

   Complete the following statement:

   If functions $f(x)$ and $g(x)$ are both ________ at $x = a$, then function ________ is also differentiable at ________, and the derivative of $f(x) + g(x)$ at $x = a$ is equal to the ________ of the ________ of ________ and ________ at ________.

3. Find the derivative of each of the following functions.

   (a) $f(x) = x^3 - 5x^2 + 7$

   (b) $g(x) = x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \cdots + \dfrac{x^{100}}{100}$

   (c) $h(x) = 375 - 300(0.96)^x$

   (d) $k(x) = 2^x \cdot 3^x + (x^3 + x)^2$

4. Each of the following represents a derivative. Use this information to evaluate each limit by inspection:

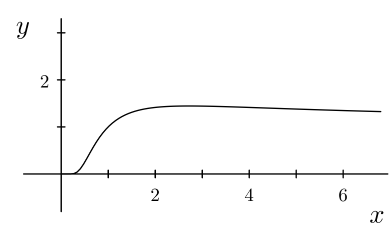
   (a) $\displaystyle\lim_{a\to 0} \frac{(x+a)^7 - a^7}{a}$

   (b) $\displaystyle\lim_{d\to 1} \frac{\ln b}{b-1}$

   (c) $\displaystyle\lim_{x\to c} \frac{2^x - 2^c}{x-c}$

   (d) $\displaystyle\lim_{d\to x} \frac{x^n - d^n}{x-d}$?

5. It is a fact that the square root of 2 is the same as the fourth root of 4 — in other words, $2^{1/2} = 4^{1/4}$. Thus the graph of $y = x^{1/x}$ goes through two points that have the same y-coordinate. Because $y = x^{1/x}$ is a *continuous* function, it is likely that there is either a *local* maximum or a *local* minimum in between $x = 2$ and $x = 4$. Why does this make sense?

   As the diagram suggests, the maximum y-value for this curve occurs between $x = 2$ and $x = 4$. Determine this is maximum value.