

Lectures on Challenging Mathematics

Calculus (part 1)

Summer 2019

Zuming Feng

Phillips Exeter Academy and IDEA Math

zfeng@exeter.edu

©Copyright 2008 – 2019 Idea Math

Contents

©Copyright 2008 – 2019 Idea Math

1	Limits and approximations	3
1.1	Algebraic knowledge and tools (part 1)	3
1.2	Limits and approximations (part 1)	4
1.3	Algebraic knowledge and tools (part 2)	5
1.4	Limits and approximations (part 2)	6
1.5	Algebraic knowledge and tools (part 3)	7
1.6	Limits and approximations (part 3)	9
1.7	Algebraic knowledge and tools (part 4)	10
1.8	Limits and approximations (part 4)	11
1.9	Algebraic knowledge and tools (part 5)	12
1.10	Limits and approximations (part 5)	13
1.11	Algebraic knowledge and tools (part 6)	14
1.12	Limits and approximations (part 6)	15
1.13	Algebraic knowledge and tools (part 7)	17
1.14	Limits and approximations (part 7)	18
1.15	Algebraic knowledge and tools (part 8)	19
1.16	Limits and approximations (part 8)	20
1.17	Algebraic knowledge and tools (part 9)	22
1.18	Limits and approximations (part 9)	23
1.19	Algebraic knowledge and tools (part 10)	24
1.20	Limits and approximations (part 10)	25
2	Derivatives	27
2.1	Algebraic knowledge and tools (part 11)	27
2.2	Limits, slopes, and derivatives (part 1)	29
2.3	Algebraic knowledge and tools (part 12)	31
2.4	Limits, slopes, and derivatives (part 2)	32
2.5	Algebraic knowledge and tools (part 13)	33
2.6	Limits, slopes, and derivatives (part 3)	34
2.7	Algebraic knowledge and tools (part 14)	35
2.8	Limits, slopes, and derivatives (part 4)	36
2.9	Algebraic knowledge and tools (part 15)	37

2.10	Limits, slopes, and derivatives (part 5)	38
2.11	Algebraic knowledge and tools (part 16)	40
2.12	Limits, slopes, and derivatives (part 6)	41
2.13	Algebraic knowledge and tools (part 17)	42
2.14	Limits, slopes, and derivatives (part 7)	43
2.15	Algebraic knowledge and tools (part 18)	44
2.16	Limits, slopes, and derivatives (part 8)	45
2.17	Algebraic knowledge and tools (part 19)	46
2.18	Limits, slopes, and derivatives (part 9)	47
2.19	Algebraic knowledge and tools (part 20)	48
2.20	Limits, slopes, and derivatives (part 10)	49
3	Operational rules for derivatives	51
3.1	Algebraic knowledge and tools (part 21)	51
3.2	Operational rules for derivatives (part 1)	52
3.3	Introduction to differential equations (part 1)	53
3.4	Operational rules for derivatives (part 2)	54
3.5	Algebraic knowledge and tools (part 22)	55
3.6	Operational rules for derivatives (part 3)	56
3.7	Introduction to differential equations (part 2)	57
3.8	Operational rules for derivatives (part 4)	58
3.9	Algebraic knowledge and tools (part 23)	59
3.10	Operational rules for derivatives (part 5)	60
3.11	Introduction to differential equations (part 3)	61
3.12	Operational rules for derivatives (part 6)	62
3.13	Algebraic knowledge and tools (part 24)	63
3.14	Operational rules for derivatives (part 7)	64
3.15	Introduction to differential equations (part 4)	65
3.16	Operational rules for derivatives (part 8)	66
3.17	Algebraic knowledge and tools (part 25)	67
3.18	Operational rules for derivatives (part 9)	68
3.19	Introduction to differential equations (part 5)	69
3.20	Operational rules for derivatives (part 10)	70
4	Properties of derivatives	71
4.1	Algebraic knowledge and tools (part 26)	71
4.2	Properties of derivatives (part 1)	73
4.3	Introduction to differential equations (part 6)	74
4.4	Properties of derivatives (part 2)	76
4.5	Algebraic knowledge and tools (part 27)	77
4.6	Properties of derivatives (part 3)	78
4.7	Introduction to differential equations (part 7)	79
4.8	Properties of derivatives (part 4)	81

4.9	Algebraic knowledge and tools (part 28)	82
4.10	Properties of derivatives (part 5)	84
4.11	Introduction to differential equations (part 8)	85
4.12	Properties of derivatives (part 6)	86
4.13	Algebraic knowledge and tools (part 29)	87
4.14	Properties of derivatives (part 7)	88
4.15	Introduction to differential equations (part 9)	89
4.16	Properties of derivatives (part 8)	90
4.17	Algebraic knowledge and tools (part 30)	91
4.18	Properties of derivatives (part 9)	93
4.19	Introduction to differential equations (part 10)	94
4.20	Properties of derivatives (part 10)	95
5	The first short review and extension	97
5.1	Algebraic knowledge and tools (part 31)	97
5.2	Properties of derivatives (part 11)	98
5.3	Algebraic knowledge and tools (part 32)	99
5.4	Properties of derivatives (part 12)	100
5.5	Algebraic knowledge and tools (part 33)	102
5.6	Properties of derivatives (part 13)	103
5.7	Algebraic knowledge and tools (part 34)	104
5.8	Properties of derivatives (part 14)	105
5.9	Algebraic knowledge and tools (part 35)	106
5.10	Properties of derivatives (part 15)	107

1.5 Algebraic knowledge and tools (part 3)

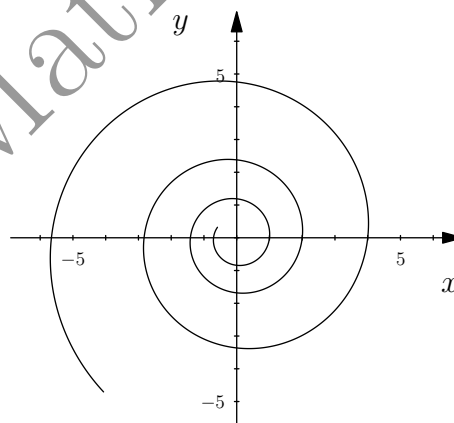
- Given that the x -intercepts of the graph of $y = f(x)$ are -3 , 2 and 7 , find an example of $f(x)$.
- If $f(x)$ is an odd function and $g(x)$ is an even function, what can be said about the following functions.

(a) $f(f(x))$ (b) $g(g(x))$ (c) $f(g(x))$ (d) $g(f(x))$

- Working in radian mode, determine all θ , with $0 \leq \theta < 2\pi$, such that

(a) $\sin(2\theta) < \sin \theta$ (b) $\sin(2\theta) = 2 \sin \theta$ (c) $\sin(2\theta) \geq \sin \theta$

- It is awkward to describe fundamental curves like *spirals* using only the Cartesian coordinates x and y . The example shown, on the other hand, is easily described with polar coordinates – all its points fit the equation $r = 2^{\theta/360}$ (using degree mode). Choose three specific points in the diagram and make calculations that confirm this. What range of θ -values does the graph represent? Use a graphing tool to obtain pictures of this spiral.



- A function f defined on a subset of the real numbers with real values is (*monotonically*) *nonincreasing* or (*monotonically*) *decreasing* on interval $[a, b]$ if $f(x) \geq f(y)$ for $x \leq y$. We say function is *strictly decreasing* on $[a, b]$ if $f(x) > f(y)$ for $x < y$. Likewise, we can define *nondecreasing* (or *increasing* and *strictly increasing* functions. We call a function *monotonic* if it is either entirely nonincreasing or nondecreasing on its domain. We call a function *strictly monotonic* if it is either entirely increasing or decreasing on its domain.

Find a function with domain \mathbb{R} (the set of real numbers) which is

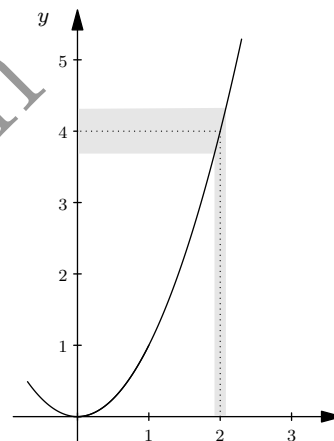
- a strictly increasing exponential function;
- a strictly decreasing exponential function;
- a strictly increasing logarithm function;
- a strictly decreasing logarithm function;
- a strictly increasing power function;
- a strictly decreasing power function;
- both decreasing and increasing (that is, both nonincreasing and nondecreasing);

1.18 Limits and approximations (part 9)

1. *One-sided limits.* Consider the *sign function*, which is defined for all nonzero values of x by $\operatorname{sgn}(x) = \frac{x}{|x|}$. Confirm that both $\lim_{x \rightarrow 0^-} \operatorname{sgn}(x)$ and $\lim_{x \rightarrow 0^+} \operatorname{sgn}(x)$ exist, and notice that they have different values. Because the two one-sided limits do not agree, $\operatorname{sgn}(x)$ does not approach a (two-sided) limit as $x \rightarrow 0$. (Notice that $\operatorname{sgn}(0)$ remains undefined, hence it is *discontinuous* at $x = 0$.)

2. Verify that $|x^3 - 8| < 0.0001$ is true for all numbers x that are sufficiently close to 2. In other words, show that the inequality is satisfied if the distance from 2 to x is small enough. How small is “small enough”?

What does “limit” mean? Let p be any small positive number. Show that there is another small positive number d , which depends on p , that has the following property: whenever $|x - 2| < d$ it is true, it is also true that $|x^3 - 8| < p$. This means —intuitively— that x^3 can be brought arbitrarily close to 8 by making x suitably close to 2. It is customary to summarize this situation by writing $\lim_{x \rightarrow 2} x^3 = 8$.



3. One might think that the concept of limit is simple: It is just about substituting the x value. For example, it is routine to find the following limits. Do so.

(a) $\lim_{a \rightarrow 2} \frac{2a^2 - 3a - 2}{2a + 1}$

(b) $\lim_{b \rightarrow 3} \frac{\sqrt{b^3 - 8}}{\sqrt{b^2 - 4}}$

(c) $\lim_{c \rightarrow \frac{\pi}{2}} \frac{2c}{\sin c}$

(d) $\lim_{d \rightarrow \pi} \frac{d}{\sin \frac{d}{2}}$

4. (Continuation) The above limits are easy to find only because we are working with functions that are *continuous* at the points of interest. Compute the following limits instead. These limits are all of *indeterminate form*. (Note that we emphasize the term “*form*” rather than the term “*value*”, because we can determine the values of these limits.)

(a) $\lim_{a \rightarrow -0.5} \frac{2a^2 - 3a - 2}{2a + 1}$

(b) $\lim_{b \rightarrow 2} \frac{\sqrt{b^3 - 8}}{\sqrt{b^2 - 4}}$

(c) $\lim_{c \rightarrow 0} \frac{2c}{\sin c}$

(d) $\lim_{d \rightarrow 0} \frac{d}{\sin \frac{d}{2}}$

(e) $\lim_{e \rightarrow -0.5} \frac{1 + e}{(2e + 1)^2}$

(f) $\lim_{f \rightarrow -0.5} \frac{1 + f}{2f + 1}$

5. Consider two positive sequences (a_1, a_2, \dots) and (b_1, b_2, \dots) . Explain why statements like “ a_n is very close to b_n ” or “ a_n and b_n are approximately the same” are very vague.

Compare the following expressions. How do they differ from each other?

(a) $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$

(b) $\lim_{n \rightarrow \infty} (a_n - b_n) = 0$

(c) $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1$

2.20 Limits, slopes, and derivatives (part 10)

- Riding a train that is traveling at 72 mph, Morgan walks at 4 mph toward the front of the train, in search of the snack bar. How fast is Morgan traveling, *relative to the ground*?
- (Continuation) Given *differentiable* functions f and g , let $k(t) = f(t) + g(t)$. Use the definition of the derivative to show that $k'(t) = f'(t) + g'(t)$ must hold. This justifies *term-by-term differentiation*.

Complete the following statement:

If functions $f(x)$ and $g(x)$ are both _____ at $x = a$, then function _____ is also differentiable at _____, and the derivative of $f(x) + g(x)$ at $x = a$ is equal to the _____ of the _____ of _____ and _____ at _____.

- Find the derivative of each of the following functions.

(a) $f(x) = x^3 - 5x^2 + 7$

(b) $g(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^{100}}{100}$

(c) $h(x) = 375 - 300(0.96)^x$

(d) $k(x) = 2^x \cdot 3^x + (x^3 + x)^2$

- Each of the following represents a derivative. Use this information to evaluate each limit by inspection:

(a) $\lim_{a \rightarrow 0} \frac{(x+a)^7 - a^7}{a}$

(b) $\lim_{d \rightarrow 1} \frac{\ln b}{b-1}$

(c) $\lim_{x \rightarrow c} \frac{2^x - 2^c}{x - c}$

(d) $\lim_{d \rightarrow x} \frac{x^n - d^n}{x - d}$?

- It is a fact that the square root of 2 is the same as the fourth root of 4 — in other words, $2^{1/2} = 4^{1/4}$. Thus the graph of $y = x^{1/x}$ goes through two points that have the same y -coordinate. Because $y = x^{1/x}$ is a *continuous* function, it is likely that there is either a *local* maximum or a *local* minimum in between $x = 2$ and $x = 4$. Why does this make sense?

As the diagram suggests, the maximum y -value for this curve occurs between $x = 2$ and $x = 4$. Determine this maximum value.

