1.4 Practices in geometry computations (part 4)

1.4.1 Mixed exercises 4

1. Two circles of radius 2 are centered at $(2, 0)$ and $(0, 2)$. What is the area of the intersection of the interiors of the two circles?

2. Let $ABCDEF$ and $ABMNOP$ be two connected regular hexagons of side length 1. Compute the area of quadrilateral $PNCE$.

3. In triangle ABC, $AB = BC = 1$ and $AC = \sqrt{2}$. Segment AB is a diameter of circle ω. Semicircle γ has its center lying on side BC. Given that γ is tangent to ω and γ passes through C, find the radius of γ.

4. In rectangle $ABCD$, $AB = 6$, $AD = 30$, and G is the midpoint of side AD. Segment AB is extended 2 units beyond B to point E, and F is the intersection of segments ED and BC. What is the area of $BFDG$?

5. What is the radius of the largest circle that you can draw on graph paper that encloses $n = 1$ lattice points? How about $n = 2, 3, 4$?

1.4.2 Area, similarity, and Ceva (part 1)

6. The sides of triangle ABC are $AB = 13$, $BC = 15$, and $CA = 14$. Point P is located within the triangle so that triangle PAB has area 28, triangle PBC has area 35, and triangle PCA has area 21. If AP is extended until it meets BC at K, what is the length of BK? If CP is extended until it meets AB at L, what is the length of AL?

7. Let $A = (0, 0)$, $B = (6, 0)$, and $C = (0, 6)$. Point P lies inside the triangle.
   (a) Describe all points P such that the area of triangle ABP is equal to 9.
   (b) Describe all points P such that the area of triangle CAP is equal to 3.
   (c) Describe all points P such that the area of triangle BCP is equal to 6.
   (d) Find point P such that the area ratio between triangles ABP, BCP, CAP is $[ABP] : [BCP] : [CAP] = 3 : 2 : 1$. Find the respective ratios into which the line AP divides the side BC, the line BP divides the side CA, and the line CP divides the side AB.

8. Let P be a point inside triangle ABC. Line AP intersects side BC in D, line BP intersects side CA in E, and line CP intersects side AB in F. Suppose that $DP = 2$, $PA = 5$, $AF = 4$, $FB = 3$.
   (a) Construct point Q on line CF such that $DQ \parallel AB$. Find two pairs of similar triangles involving point Q and compute BD/CD.
   (b) Construct point R on line CF such that $BR \parallel AD$. Find two pairs of similar triangles involving point R and compute BD/CD.

9. Equilateral triangles ABE, BCF, CDG, and DAH are constructed outside the unit square $ABCD$. Eliza wants to stand inside octagon $AEBFCGDH$ so that she can see every point in the octagon without being blocked by a side of the octagon. What is the area of the region in which she can stand?

10. In triangle ABC, $\angle C = 90^\circ$, $AC = 10$, and $BC = 12$. Semicircle ω, with AC as its diameter, is erected outside of the triangle. Let P denote the midpoint of ω. Segment BP cuts the whole region (bounded by sides BA, BC, and ω) into two pieces. What is the positive difference between the areas of the two pieces?

1.9 Power-of-a-point theorem (part 2)

1.9.1 Power-of-a-point theorem (part 2)

1. In quadrilateral $ABED$, rays AD and BE meet at C. Suppose that $\angle ABE = \angle CDE$. Given that $DA = 2$, $CE = 4$, $EB = 8$, and $CD = x$. Find x by showing that $x(x+2) = 4 \cdot 12$.

   Note that $x = -8$ is a also an solution to $x(x+2) = 4 \cdot 12$. We are trying to find a configuration to explain this fact. Consider a self-intersecting quadrilateral $ABED$ with sides AB and DE intersecting each other. We also consider the concept of *directed length*. In particular, assigne one direction to CA and CD, and the opposite direction to DA; that is, $DA = -AD$. Explain the rest of the details.

2. Let ω be a circle with center O and radius R, and let X be a point in the plane that is outside of ω. A line is drawn through X; it intersects ω at Y and Z. (If the line is tangent to the circle, then $Y = Z$). Prove that $XY \cdot XZ = XO^2 - R^2$ by either

   (a) constructing the line XT that is tangent to ω at T; or

   (b) constructing the line XO.

3. (Continuation) A similar result can be established when X is inside the circle. State and prove this result. Sometimes, this is called the *cross-chord theorem*.

4. Putting the results in the previous two problems together together, we have the following statement:

   > Let ω be a circle with center O and radius R, and let X be a point in the plane. A line is drawn through X; it intersects ω at Y and Z. (If the line is tangent to the circle, then $Y = Z$). Then the product $XY \cdot XZ = XO^2 - R^2$ is constant (it does not depend on the line drawn), and it is called the *power* of X with respect to ω. (The lengths are directed.)

   (a) Given the circle ω in the plane, determine the respective regions of all points in the plane that has positive, negative, zero power respect to ω.

   (b) Restate a simplified version of the statement by complete the following sentence.

   > Let ω be a circle, and let X be a point in the plane. Two lines are drawn through X. If one line intersects ω at Y and Z and the other line intersects ω at P and Q, then ____________________.

   (c) State the converse of the statement. Is the converse statement true? One needs to be a bit careful about configurations. (Hint:
   This inverse together with the original statement is called the *Power-of-a-point theorem*.

5. Find all points Q in the coordinate plane such that there are exactly two circles passing through points $P = (2, 3)$ and Q, both tangent to the x-axis.

1.9.2 Tangent circles

6. We are given two lines that are perpendicular to each other. A circular disc of radius 1 is placed so that it touches both of these lines. A larger circular disc of radius r is then placed so that it touches the smaller disc and both of these lines. Compute r.

7. Let XYZ be a triangle with $\angle XYZ = 40^\circ$ and $\angle YZX = 60^\circ$. A circle ω, centered at the point I, lies inside triangle XYZ and is tangent to all three sides of the triangle. Let A be the point of tangency of ω with side YZ, and let ray XI intersect side YZ at B. Determine the measure of $\angle AIB$.

8. A square region $ABCD$ is externally tangent to the circle with equation $x^2 + y^2 = 1$ at the point $(0, 1)$ on the side CD. Vertices A and B are on the circle with equation $x^2 + y^2 = 4$. What is the side length of this square?

9. Circles with centers A and B have radii 1 and 2, respectively. The distance between centers of circles is 6. Find the radius of a circle that is tangent to both of these circles and to the segment AB.

10. Circles of radii $5, 5$ and 8 are mutually externally tangent to each other. A smaller circle with radius m/n, where m and n are relatively prime positive integers, is placed so that it is externally tangent to all of these circles. Find $m + n$.

2.4 Challenges in geometry calculations (part 2)

1. Let A, B, C, D be four points, arranged in clockwise order, on circle ω. Segments AC and BD intersect at P. Given that $AB = 3$, $BP = 4$, $PA = 5$, $PC = 6$, find the radius of the circle ω.

2. The 8×18 rectangle $ABCD$ is cut into two congruent hexagons in such a way that the two hexagons can be repositioned without overlapping to form a square. What is the perimeter of the hexagon?

3. Let $ABCD$ be a trapezoid with $AB \parallel CD$ and $CD = 2AB = 2AD$. Suppose that $BD = 6$ and $BC = 4$, find the area of the trapezoid.

4. Circle ω, centered at I, is inscribed in quadrilateral $ABCD$. Given that $AB = 4$, $BC = 6$, $DA = 3$, $DB = 5$. Find AI.

5. In triangle ABC, $AB = 4$ and $AC = 5$. Point D lies on side AB with $AD = 3$. Let X be a point on side AC. Segments BX and CD meet in P. Compute DP/PC and BP/PX for each of the following cases.

   (a) $AX = XC$ (b) $\angle ABX = \angle XBC$

2.5 Geometry project 3: Folding, unfolding, and 3-D visions

1. The height of a cylindrical pole is 12 feet and its circumference is 2 feet. A rope is attached to a point on the circumference at the bottom of the pole. The rope is then wrapped tightly around the pole four times before it reaches a point on the top directly above the starting point at the bottom. What is the minimum number of feet in the length of the rope? Express your answer in simplest radical form.

2. A θ-degree sector of a circle of radius r is folded to a right cone by aligning the two straight sides. Compute the volume of the cones.

   (a) $\theta = 180°, r = 12$ (b) $\theta = 216°, r = 10$ (c) $\theta = 288°, r = 10$

3. Consider the paper rectangle $ABCD$. Point E lies on side AB and points F and G lie on side CD. Construct segments AG, GE, EF, FB. Given that $AG = GE = EF = FB = AE = EB = FG = 5$. One can fold this rectangular piece of paper along constructed segment to obtain a regular tetrahedron. What is the volume of the tetrahedron?

4. A cone-shaped mountain has its base on the ocean floor and has a height of 8000 feet. The top $\frac{1}{8}$ of the volume of the mountain is above water. What is the depth of the ocean at the base of the mountain, in feet?

5. A sphere of radius is inscribed in a cone with a base of 6. What is the height of the cone? What is the surface area (without the base) of the cone?