1.3 Arithmetic and geometric progressions (part 2)

1. Call a 3-digit number *geometric* if it has 3 distinct digits which, when read from left to right, form a geometric sequence. Find the difference between the largest and smallest geometric numbers.

2. When asked to evaluate the sum of an arithmetic series, Blair claimed "That's easy – just take average the first and last terms, then multiply by the number of terms." Explain Blair's strategy.

   Consider now the typical geometric series, which looks like

   $$\text{first} + (\text{first} + \text{difference}) + (\text{first} + 2 \cdot \text{difference}) + \cdots + \text{last}.$$

   Find a compact formula for this arithmetic series in terms of *first*, *last*, and *difference*.

3. Set S consists of all the positive integers less than or equal to 2016. Set A consists of all the numbers in S that have remainder 1 or 2 when divided by 6. Set B consists of all the numbers in S that have remainder 4 or 5 when divided by 6. Set C consists of all the remaining numbers of set S. Find each of the following.

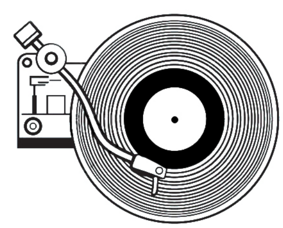
   (a) The numbers of elements in each of A, B, C.

   (b) The sum of elements in C.

   (c) The sum of elements in A.

   (d) The difference between the sum of the elements in A and the sum of the elements in B.

4. A typical long-playing phonograph record (once known as an LP) plays for about 24 minutes at 33 1/3 revolutions per minute while a needle traces the long groove that spirals slowly in towards the center. The needle starts 5.7 inches from the center and finishes 2.5 inches from the center. Estimate the length of the groove. Why it is an *estimation* rather than an accurate answer? In other words, what assumptions did you make in answering this question?



5. A triangle with a 32-inch side, a 40-inch side, and a 50-inch side is a curiosity, for its sides form a geometric sequence. Find the constant multiplier for this sequence. Find other such triangles. Are there any restrictions on the multipliers that can be used?

1.14 Revisit quadratic formula (part 2)

1. Find real numbers x and y such that $x^2 + 12xy + 52y^2 - 8y + 1 = 0$.

2. Consider $g(x, y) = x^2 + 8xy + 25y^2 - 12y + 4$.

   (a) Find all extreme values (that is, either a maximum or a minimum value) of $g(x, 5)$. Find all extreme values of $g(-1, y)$.

   (b) Suppose y is a fixed number. Find all extreme values of $g(x, y)$. Your answers should be in terms of y, but not x.

   (c) Determine the minimum value of $g(x, y)$.

   (d) Solve the equation $g(x, y) = 0$.

3. Let c be the larger solution to the equation $x^2 - 20x + 13 = 0$. Compute the area of the circle with center (c, c) passing through the point $(13, 7)$.

4. Find all ordered pairs of real numbers (x, y) such that $x^2y = 3$ and $x + xy = 4$.

5. Determine the minimum value of $y = x^4 + 4x + 1$ (by completing the square *twice*.)

1.15 Recursive relations (part 2)

1. In how many ways can one choose a team of k people from the n people sitting in a room?

   You are probably familiar with binomial coefficients; the answer to this problem is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. There is a nice recursion satisfied by these coefficients:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

   Can you find a combinatorial proof of this fact?

2. Prove, using Pascal's recursion, that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

3. Find recursive and explicit formula that describes the sequence of triangular numbers 1, 3, 6, 10, 15, ... Why are these numbers called *triangular numbers*? Explain why the sum of two consecutive triangular numbers is a perfect square. (Note that this fact can be explained algebraically and geometrically.)

4. Choose a positive value for x_0 that no one else will think of, then calculate seven more terms of the sequence defined recursively by

$$x_n = \frac{1 + x_{n-1}}{1 - x_{n-1}}.$$

   What do you notice?

5. Given a 2×10 grid, determine the number of domino (1×2 grid) tilings of the board.

1.16 Operations rules with logarithm (part 5)

1. Without calculator, find x:

   (a) $\log_4 x = -1.5$
   (b) $\log_x 8 = 6$
   (c) $27 = 8(x-2)^3$
   (d) $3\log_{27}(x-2)^4 = 2$

2. Rewrite the equation $19\log x - 9 + 9\log y = 199\log 8z$ so that it makes no reference to logarithms.

3. Earthquakes can be classified by the amount of energy they release. Because of the large numbers involved, this is usually done logarithmically. The Richter scale is defined by the equation $R = 0.67\log(E) - 1.17$, where R is the rating and E is the energy carried by the seismic wave, measured in kilowatt-hours. (A kilowatt-hour is the energy consumed by ten 100-watt light bulbs in an hour).

   (a) The 1989 earthquake in San Francisco was rated at 7.1. What amount of energy did this earthquake release? It could have sustained how many 100-watt light bulbs for a year?

   (b) An earthquake rated at 8.1 releases more energy than an earthquake rated at 7.1. How many times more?

   (c) Rewrite the defining equation so that E is expressed as a function of R.

   (d) Adding 1 to any rating corresponds to multiplying the energy by what constant?

   (e) Is it possible for a seismic wave to have a *negative* rating? What would that signify?

4. When $10^{3.43429448}$ is evaluated, how many digits are found to the left of the decimal point? You can answer this question without using your calculator, but you *will* need it to find the first three digits. What are the first three digits when $10^{9.43429448}$ is evaluated?

5. Solve each of the following equations for x.

   (a) $2\log(2x)^3 = 3\log(x-15)^2$
   (b) $\log_2 x + \log_x 2 = \dfrac{34}{15}$