2.2 Calculating in two ways

1. A class of 10 students took a math test. Each problem was solved by exactly 7 of the students. If the first nine students each solved 4 problems, how many problems did the tenth student solve?

2. In a certain country, there are 100 senators, each of whom has 4 aides. These senators and aides serve on various committees. A committee may consist either of 5 senators, of 4 senators and 4 aides, or of 2 senators and 12 aides. Every senator serves on 5 committees, and every aide serves on 3 committees. How many committees are there altogether?

3. A 15×15 square is tiled with unit squares. Each vertex is colored either red or blue. There are 133 red points. Two of those red points are corners of the original square, and another 32 red points are on the sides. The sides of the unit squares are colored according to the following rule: If both endpoints are red, then it is colored red; if the points are both blue, then it is colored blue; if one point is red and the other is blue, then it is colored yellow. Suppose that there are 196 yellow sides. How many blue segments are there?

4. Several positive integers are given, not necessarily all different. Their sum is 2003. Suppose that n_1 of the given numbers are equal to one, n_2 of them are equal to two, and so on, n_{2003} of them are equal to 2003. Find the largest possible value of $n_2 + 2n_3 + 3n_4 + \cdots + 2002n_{2003}$.

5. Four boys (Bs) and three girls (Gs) will be seated in a row. When a boy is next to a girl, we will call this a *meeting point*. When the seven kids are seated, there may be only one meeting point, as in BBBBGGG, or there may be as many as six meeting points, as in BGBGBGB. Given all of the possible seating arrangements for these seven kids, what is the average number of meeting points per seating arrangement? Express your answer as a common fraction.

5.5 Recursive counting (part 2)

1. Let $A_1A_2\dots A_n$ ($n \ge 3$) be a regular n-sided polygon with O as its center. Triangular regions OA_iA_{i+1}, $1 \le i \le n$ (and $A_{n+1} = A_1$) are to be colored red, blue, green, or yellow such that adjacent regions are colored in different colors. Let p_n denote the number of such colorings. Express $p_{n+1} + p_n$ in closed form, and use this recursive relation to express p_n in explicit (and closed) form.

2. Consider sequences that consist entirely of `A`s and `B`s and that have the property that every run of consecutive `A`s has even length, and every run of consecutive `B`s has odd length. Examples of such sequences are `AA`, `B`, and `AABAA`, while `BBAB` is not such a sequence. How many such sequences have length 14?

3. In a game similar to three card monte, the dealer places three cards on the table: the queen of spades and two red cards. The cards are placed in a ow, and the queen starts in the center; the card configuration is thus `RQR`. The dealer proceeds to move. With each move, the dealer randomly switches the center card with one of the two edge cards (so the configuration after is the first move is either `RRQ` or `QRR`). What is the probability that, after 2004 moves, the center card is the queen?.

4. ARMLovian, the language of the fair nation of ARMLovia, consists only of words using the letters `A`, `R`, `M`, and `L`. All words can be broken up into syllables that consist of exactly one vowel, possibly surrounded by a single consonant on either or both sides. For example, `LAMAR`, `AA`, `RA`, `MAMMAL`, `MAMA`, `AMAL`, `LALA`, `MARLA`, `RALLAR`, and `AAALAAAAAMA` are ARMLovlian words, but `MRLMRLM`, `MAMMMAL`, `MMMMM`, `L`, `ARM`, `ALARM`, `LLAMA`, and `MALL` are not. Compute the number of seven-letter ARMLovian words.

5. Polyhedron Hopping!
   (a) Travis is hopping around on the vertices of a cube. Each minute he hops from the vertex he's currently on to the other vertex of an edge that he is next to. After four minutes, what is the probability that he is back where he started?
   (b) While Travis is having fun on cubes, Sherry is hopping in the same manner on an octahedron. An octahedron has six vertices and eight regular triangular faces. After five minutes, how likely is Sherry to be one edge away from where she started?
   (c) In terms of k, for $k > 0$, how likely is Travis to be back where he started after $2k$ minutes?
   (d) In terms of k, for $k > 0$, how likely is it that after k minutes Sherry is at the vertex opposite the vertex where she started?