1.5 Algebra 2.1

1. Suppose a, b, and c are positive real numbers such that $a^2 + b^2 = c^2$ and $\log a + \log b = \log c$. Compute
$$\frac{(a+b+c)(a-b+c)(a+b-c)(a-b-c)}{c^2}.$$

2. Given that $\log_{10} \sin x + \log_{10} \cos x = -1$ and that $\log_{10} (\sin x + \cos x) = \frac{1}{2} (\log_{10} n - 1)$, find n.

3. For some positive real number m, the acute angle θ formed by the lines $y = mx$ and $my = x$ is such that $\tan \theta = \frac{5}{12}$. Compute $m + 1/m$.

4. For positive numbers a, b, c, d, show that among the four equations
$$\begin{aligned} x^2 + 2\sqrt{2a+b}x + 2\sqrt{cd} &= 0, \\ x^2 + 2\sqrt{2b+c}x + 2\sqrt{da} &= 0, \\ x^2 + 2\sqrt{2c+d}x + 2\sqrt{ab} &= 0, \\ x^2 + 2\sqrt{2d+a}x + 2\sqrt{bc} &= 0, \end{aligned}$$
at least two of them each of which has two distinct real roots.

5. Graph $y = \dfrac{b^x - 5}{b^x + 3}$, assuming that the base b is greater than 1. Identify both asymptotes. Does this graph have symmetry?

1.9 Solving equations

1. For
$$x = \frac{4}{\left(\sqrt{5}+1\right)\left(\sqrt[4]{5}+1\right)\left(\sqrt[8]{5}+1\right)\left(\sqrt[16]{5}+1\right)},$$
compute $(x+1)^{48}$.

2. Solve the equation
$$\frac{x^2+x+1}{x^2+1} + \frac{2x^2+x+2}{x^2+x+1} = \frac{19}{6}.$$

3. Let $x_1 < x_2 < x_3$ be the three real roots of equation $\sqrt{2014}x^3 - 4029x^2 + 2 = 0$. Find $x_2(x_1+x_3)$.

4. Find the sum of the all the real solutions to the equation
$$\sqrt{x+2015} = x^2 - 2015.$$

5. Let $f(x) = x^2 - 2x$. How many distinct real numbers c satisfy $f(f(f(f(c)))) = 3$?

4.3 Algebra 2.4

1. Let $f(x) = x^2 + 6x + 7$. Determine the smallest possible value of $f(f(f(f(x))))$ over all real numbers x.

2. Compute the area of the region defined by $x^2 + y^2 \leq |x| + |y|$.

3. Solve the equation $(6x + 7)^2(3x + 4)(x + 1) = 6$.

4. Consider the parabola consisting of the points (x, y) in the real plane satisfying

$$x^2 - 2xy + y^2 - 4x + 2y + 3 = 0.$$

Find the minimum possible value of y.

5. The function f, defined by $f(x) = \frac{ax+b}{cx+d}$, where a, b, c, and d are nonzero real numbers, has the properties $f(19) = 19$, $f(97) = 97$, and $f(f(x)) = x$, for all values of x, except $-d/c$. Find the range of f.