1.8 Operations rules with integer exponents (part 2)

1. What is the value of $\frac{5^7}{5^7}$? of $\frac{8^3}{8^3}$? of $\frac{c^{12}}{c^{12}}$? What is the value of any number divided by itself? If you apply the common-base rule dealing with exponents and division, $\frac{5^7}{5^7}$ should equal 5 raised to what power? and $\frac{c^{12}}{c^{12}}$ should equal c raised to what power? It therefore makes sense to define c^0 to be what?

2. The result of dividing 5^7 by 5^3 is 5^4. What is the result of dividing 5^3 by 5^7, however? By considering such examples, decide what it means to put a *negative* exponent on a base.

3. Exponents are routinely encountered in science, where they help to deal with small numbers. For example, the diameter of a proton is 0.0000000000003 cm. Explain why it is logical to express this number in scientific notation as 3×10^{-13}. Calculate the surface area and the volume of a proton. Express your answers in scientific notation.

4. Write 2^{-3} in a more familiar form. How about $2^{-2} \cdot 5^{-2}$?

5. For integers m and n, we have been using the following rules of exponents:

   (i) $a^m \cdot a^n = a^{m+n}$ (ii) $\frac{a^m}{a^n} = a^{m-n}$ (iii) $a^m \cdot b^m = (ab)^m$ (iv) $\frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$

   Find two pairs of integers m and n, one with $mn > 0$ and one with $mn < 0$, to explain all of these rules.

6. (Continuation) Explain why properties (i) and (ii) are basically equivalent to each other, and so do properties (iii) and (iv).

7. (Continuation) Explain why the following properties are also natural consequences.

   (v) $(a^m)^n = a^{mn}$ (vi) $a^0 = 1$.

8. What is half of 2^{40}? What is one third of 3^{18}?

9. Find a pair of distinct positive integers a and b such that $a^b = b^a$. Then find two distinct pairs of positive integers a and b such that $a^b + b^a = 100$.

10. Explain your opinions of each of the following student responses:

    (a) Asked to find an expression equivalent to $x^9 - x^3$, a student responded x^6.

    (b) Asked to find an expression equivalent to x^9/x^3, a student responded x^3.

    (c) Asked to find an expression equivalent to $\frac{x^9 - x^6}{x^3}$, a student responded $x^6 - x^3$ and another student said $x^3 - x^2$.

(d) One student said that $\dfrac{x^3}{x^9 - x^6}$ is equivalent to $\dfrac{1}{x^6} - \dfrac{1}{x^3}$, another student said it is equivalent to $\dfrac{1}{x^3} - \dfrac{1}{x^2}$.

1.9 Solving linear equations

1. An *equation* is formed when an equal sign (=) is placed between two expressions creating a left and right side of the equation. An equation that contains one or more variables is an *open sentence*. Here are some examples,

$$3x + 1 = 10, \qquad a + 2 = a + 2, \qquad 6 - b = 5.$$

When the variable in a single-variable equation is replaced by a number, the resulting statement can be true or false. If the statement is true, the number is a *solution of an equation*.

Solve:

(a) $x - 4 = -9$ (b) $-6 = n - (-2)$ (c) $-8x = 1$ (d) $\dfrac{x}{-3} = 15$

2. Solve:

(a) $\frac{1}{3}x + 2 = -1$ (b) $3x + 5(x - 4) = 14$

(c) $4x - 3(x - 6) = 11$ (d) $2(x - 1) - 5(x + 2) = 4(x + 3)$

3. Find all mistakes in the steps of the following solutions:

(a)

$$\begin{aligned} 20x - 4 &= 6, \\ 20x - 4 + 4 &= 6 + 4, \\ 20x &= 10, \\ x &= 2. \end{aligned}$$

(b)

$$\begin{aligned} -3(y + 12) &= 51, \\ -3y + 12 &= 51, \\ -3y &= 39, \\ y &= -13. \end{aligned}$$

4. Find how many solutions each of the following equations has

(a) $0 \cdot x = 0$ (b) $4x = 0$

(c) $0 \cdot x = 5$ (d) $|x| = x$

(e) $4(x + 3) = 4x + 12$ (f) $x + 2 = x + 6$

(g) $x + 3 = 2x - 1$ (h) $|x| + 4 = 2|x| + 1$

5. A video store charges \$8 to rent a video game for one week. You must be a member to rent from the store, but the membership is free. A video game club in town charges only \$4 to rent a game for a week, but membership in the club is \$100 per year. Assume you are going to rent one video game every week for a year. Which rental plan is more economical?

6. Match each of the following problems with the equation that solves the problem.

   (a) You owe \$16 to your cousin. You paid x dollars back and you now owe \$4. How much did you pay back?

   (b) The temperature was x degrees Celsius. During a one month period it rose 16°C and is now 4°C. What was the original temperature?

   (c) A telephone pole extends 4 meters below the ground and 16 meters above the ground. What is the total length x of the pole?

   (d) A diver was swimming 16 meters below water surface. Then she saw a fish 4 meters below her. What is the fish's position with respect to the water surface?

   (i) $x - 4 = 16$ (ii) $x + 16 = 4$ (iii) $16 - x = 4$ (iv) $-16 + (-4) = x$

7. Suppose we can reduce a given equation to a linear equation of the form $ax + b = 0$, where a and b are constants. To solve it, we use one of the three possible cases:

   • If $a \neq 0$ then $x = -\dfrac{b}{a}$;

   • If $a = 0$ and $b = 0$ then x can be any number;

   • If $a = 0$ and $b \neq 0$, there are no solutions for x.

   Solve the following equations for unknown variables:

   (a) $5a + 11 = 7a - 9$ (b) $\frac{3}{7}m + 9 = 15$
   (c) $9t = -9t$ (d) $3x + 2^3 = x + 3^2$
   (e) $5 - 2(3\ell - 4) = 5\ell - 9$ (f) $2(3s + 5) = 8(13 - 21s) - 34$

8. An *identity* is an equation that is true for all acceptable values of the variables involved. Determine which of the following are identities:

   (a) $2(x - 3) + 5 = 3(x - 2) + 5$ (b) $3(x - 4) = 3x - 4$
   (c) $7(x - 3) \cdot \dfrac{1}{7} = (x - 3)$ (d) $\dfrac{y}{2} + 2 = \dfrac{1}{3}\left(y + \dfrac{y}{2} + 6\right)$

9. In a family, the father is 41 years old and his son is 14 years old. In how many years will the father be twice as old as his son?

   Although you may be able to solve this problem using a method of your own, try the following approach, which starts by guessing the correct answer. Study the first two rows of the table below, which is based on 10 year and 20 year guesses for the answer. What can you conclude from this information? Then make your own guess and use it to fill in the next row of the table. If you have not guessed the correct answer, use another row of the table and try again.

Guess	Farther's age in the future	Son's age in the future	Target	Check
10	$41 + 10 = 51$	$14 + 10 = 24$	$51 - 2 \cdot 24 = 3$	no
20	$41 + 20 = 61$	$14 + 20 = 34$	$61 - 2 \cdot 34 = -7$	no

10. Let (a, b, c, d, e, f) be an arithmetic sequence in this order.

    (a) If $b - a = k$, express each of c, d, e, f in terms of a and k. Assuming that $a = 2015$ and $d = 2020$, find k, b, c, e, f.

    (b) If $f - e = \ell$, express each of a, b, c, d in terms of f and ℓ. If $c = 2010$ and $f = 2015$, evaluate k, a, b, d, e.

1.10 Evaluating algebraic expressions

1. Evaluate each of the following.

   (a) $2x + y + 1$ for $x = 3$, $y = -5$

   (b) $5m - \dfrac{n}{3}$ for $(m, n) = (-4, -21)$

   (c) $8x - 4y - 2z$ for $(x, y, z) = \left(\dfrac{1}{2}, -\dfrac{1}{4}, \dfrac{3}{2}\right)$

   (d) $|p| + |q|$ for $p = -31$, $q = -12$

   (e) $|p + q|$ for $p = -31$, $q = 12$

   (f) $|p - q|$ for $(p, q) = (-31, -12)$

2. In baseball statistics, a player's slugging ratio is defined to be $\dfrac{s + 2d + 3t + 4h}{b}$, where s is the number of singles, d the number of doubles, t the number of triples and h the number of home runs obtained in b times at bat. Dana came to bat 75 times during the season, and hit 12 singles, 4 doubles, 2 triples, and 8 home runs. What is Dana's slugging ratio, rounded to three decimal places?

3. Without using parentheses, write an expression equivalent to $3(4(3x - 6) - 2(2x + 1))$.

4. Given that $m = 25q + 10d + 5n + c$, find the value of m when $q = 3$, $d = 5$, $n = 7$, $c = 11$. Make up a word problem to go with the equation $25q + 10d + 5n + c = 100$.

5. Determine the number of quadruples (q, d, n, c) of positive integers such that $25q + 10d + 5n + c = 100$.

6. Let $s = 60m$ and $m = 60h$.

   (a) Express s in terms of h and compute s for $h = 1$, $h = 2$, and $h = 5.7$, respectively.

   (b) Express h in terms of s and compute h for $s = 2400$ and $s = 100$ respectively.

   Can you find real-world meanings for these variables?

7. Temperature is measured in both Celsius and Fahrenheit degrees. These two systems are of course related: the *Fahrenheit* temperature is obtained by adding 32 to 9/5 of the *Celsius* temperature. In the following questions, let C represent the Celsius temperature and F the Fahrenheit temperature.

   (a) In Fahrenheit, the respective average monthly temperatures in Exeter (New Hampshire, USA) for January, April, July, and October are $10°, 33°, 58°$, and $36°$. Convert those temperatures into Celsius.

   (b) In Celsius, the respective average temperatures in Riyadh (Saudi Arabia) for January, April, July, and October are $15°$, $27°$, $36°$, and $28°$. Convert those temperatures into Fahrenheit.

   (c) Write an equation that expresses F in terms of C.

   (d) Use this equation to find the value of F that corresponds to $C = 20$.

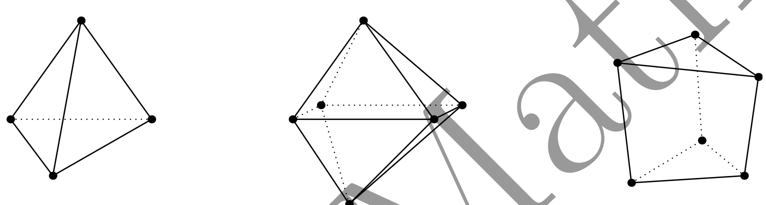
   (e) On the Celsius scale, water freezes at $0°$ and boils at $100°$. Use your formula to find the corresponding temperatures on the Fahrenheit scale. Do you recognize your answers?

(f) For what value(s) the temperature is the same in both systems.

8. (Continuation) A quick way to get an approximate Fahrenheit temperature from a Celsius temperature is to double the Celsius temperature and add 30. Explain why this is a good approximation. Convert 23° Celsius the quick way. What is the difference between your answer and the correct value? For what Celsius temperature does the quick way give the correct value?

9. Evaluate $|3 - |6 - x||$ for each of $x = \pi$, 2π, 3π.

10. For a 3-dimensional object, let v denote the number of its vertices, e denote the number of its edges, and f denote the number of its faces. For each of the following object, compute the value $v - e + f$. (For example, for a cube, we have $v = 8$, $e = 12$, $f = 6$, and $v - e + f = 2$.)



(a) A tetrahedron. (Shown in the left-hand side figure above.)

(b) An octahedron. (Shown in the middle figure above.)

(c) A triangular prism. (Shown in the right-hand side figure above.)

(d) The solid obtained in the following way: Gluing 27 unit cubes together to form a $3 \times 3 \times 3$ cube, and removing 8 unit cubes one from each corner.

1.11 Linear equations and problem solving (part 1)

1. Solve the equations for unknown variables:

   (a) $\frac{1}{3}h + \frac{1}{4} = h - \frac{1}{6}$

   (b) $\frac{x-1}{2} + \frac{x-2}{3} = 5 - \frac{x-3}{4}$

2. Solve:

   (a) $7t - 5t = 2t$

   (b) $91 - 7|x-5| = 0$

   (c) $3x + 4^5 = 5x + 4^3$

   (d) $4 - 3|x+2| = 0$

3. The length of a certain rectangle exceeds its width by exactly 8 cm, and the perimeter of the rectangle is 66 cm. What is the width of the rectangle? Although you may be able to solve this problem using a method of your own, try the following approach, which starts by guessing the width of the rectangle. Study the first row of the table below, which is based on a 10 cm guess for the width. Then make your own guess and use it to fill in the next row of the table. If you have not guessed the correct width, use another row of the table and try again.

Guess	Length	Perimeter	Target	Check
10	10 + 8 = 18	2(10) + 2(18) = 56	66	no

4. Solve the following equations and check your result by plugging it back in:

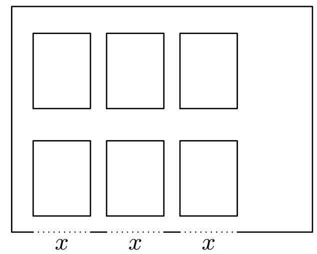
   (a) $9(a-4) = 5(3a-2)$

   (b) $\frac{2}{3}(9x+6) = 13 - (1-2x)$

   (c) $\frac{x}{5} = \frac{x}{7} + \frac{x}{11}$

   (d) $\frac{|x|}{3} = \frac{2|x|}{5} + \frac{4}{7}$

5. Norman High is in a city, and West Lake High is one of the city's suburbs. Norman High's enrollment has been decreasing at an average rate of 75 students per year, whereas West Lake High's enrollment has been increasing at an average rate of 60 students per year. Norman High has currently 3150 students, and West Lake High has 2475. If enrollments continue to change at the same rates, when will the two schools have the same number of students?



6. A page of your school yearbook is 21 cm in length and 28 cm in width. The left margin is 2 cm and the space to the right of the pictures is 7 cm. The space between pictures is 1.5 cm. How wide can each picture be to fit three across the width of the page?

7. At East High School, 579 students take Spanish. This number has been increasing at a rate of about 30 students per year. The number of students taking French is 217 and has been decreasing at a rate of about 2 students per year. At these rates, when will there be three times as many students taking Spanish as taking French?

8. Sammy felt very generous. He gave Becky half his pennies. Then, an hour later, Sammy gave Janie one fourth of his remaining pennies. Shortly after, Mary borrowed half the pennies that Sammy had left. Sammy then had 12 pennies. How many pennies did Sammy give to Becky?

9. Three water pipes are used to fill a swimming pool. Used alone, the first pipe takes 8 hours, the second pipe takes 12 hours, and the third pipe takes 24 hours. If all three pipes are used together, how many hours will take to fill the pool?

10. A gazelle can run 73 feet per second for several minutes. A cheetah can run faster (88 feet per second) but can only sustain its top speed for about 20 seconds before it is worn out. How far away from the cheetah does the gazelle need to stay for it to be safe?