3.5 MO2M2 practice test 5

1. For nonnegative real numbers a, b, c, prove that

$$a^4b^2 + b^4c^2 + c^4a^2 + 3a^2b^2c^2 \geq abc(a^2b + b^2a + b^2c + c^2b + c^2a + a^2c).$$

2. Given real numbers $x_1, x_2, \ldots, x_n$ with $\sum_{i=1}^{n} |x_i| = 1$ and $\sum_{i=1}^{n} x_i = 0$, prove that

$$\left|\sum_{i=1}^{n} \frac{x_i}{i}\right| \leq \frac{1}{2} - \frac{1}{2n}.$$

3. Find all positive integers $n > 1$ such that the sequence $2n - 1, 3n - 1, 4n - 1, 5n - 1, \ldots$ contains a perfect k^{th} power for every positive integer k.

4. Determine whether it is possible to place the integers $1, 2, \ldots, 2012$ in a circle in such a way that the 2012 products of adjacent pairs of numbers leave pairwise distinct remainders when divided by 2013.

5. Let ABC be an acute triangle. Let ω be a circle whose center L lies on side BC and is tangent to side AB and AC at B_1 and C_1, respectively. Suppose that the circumcenter of triangle ABC lies on the minor arc $\widehat{B_1C_1}$ of ω. Prove that the circumcircle of ABC and ω meet at two points.