1.16 Inequality practice set 3

1. Given that a, b, c, x, y, z are real numbers satisfying the conditions

$$a^2 + b^2 + c^2 = 25, \quad x^2 + y^2 + z^2 = 36, \quad ax + by + cz = 30,$$

evaluate $\dfrac{a+b+c}{x+y+z}$.

2. Prove that for all real numbers a, b, c we have

$$(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \le (a^2 + b^2 + c^2)(a^2b^2 + b^2c^2 + c^2a^2).$$

3. For positive real numbers a, b, c prove that

$$2\sqrt{bc + ca + ab} \le \sqrt{3}\sqrt[3]{(b+c)(c+a)(a+b)}.$$

4. Let θ be an acute angle. Prove that if x and y are positive real numbers with

$$\frac{x^3}{\cos\theta} + \frac{y^3}{\sin\theta} = 1,$$

then $x^2 + y^2 \le 1$.

5. For positive real numbers a, b, c, d with $ab + bc + cd + da = 1$, prove that

$$\frac{a^3}{b+c+d} + \frac{b^3}{c+d+a} + \frac{c^3}{d+a+b} + \frac{d^3}{a+b+c} \ge \frac{1}{3}.$$

1.17 Algebra practice set 6

1. If a, b, c are distinct real numbers, simplify completely

$$\frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)} + \frac{(x-a)(x-b)}{(c-a)(c-b)}.$$

   Hmm ... If you think more carefully, can you predict the answer?

2. Prove that for any positive integers a, b, and n we have

$$\left\lfloor \frac{\lfloor \frac{n}{a} \rfloor}{b} \right\rfloor = \left\lfloor \frac{n}{ab} \right\rfloor,$$

   where $\lfloor x \rfloor$ is equal to the largest integer less than or equal to x.)

3. Let $f(n)$ be a function such that $f(n) = \left\lceil \frac{n}{2} \right\rceil + f\left(\left\lceil \frac{n}{2} \right\rceil\right)$ for every integer n greater than 1. If $f(1) = 1$, find the maximum value of $f(k) - k$, where k is a positive integer less than or equal to 2011. (For real number x, $\lceil x \rceil$ is equal to the smallest integer larger than or equal to x.)

4. In the coordinate plane, let $\mathcal{R}$ denote the region consisting of points (x, y) such that

$$3(|x| + |y| - 4)^2 + 4|xy| \le 12$$

   What is the area of region $\mathcal{R}$?

5. For irrational numbers a and b with $a > 0$, determine if a^b is always irrational.

1.18 Introduction to functional equations (part 3)

1. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(2x) = 2f(x)$ and $|x - f(x)| \leq 1$ for all x in $\mathbb{R}$.

2. Identify all asymptotic behavior of the graph

$$g(x) = \frac{x^4 - x^3 + 7x^2 - x + 1}{x^3 + x}.$$

   Determine, with justification, if this graph has a half-turn symmetry.

3. Determine if the following statement is true.

   If $f : \mathbb{R} \to \mathbb{R}$ is a function for which $f(x + 10) = f(x) + f(10)$, then f is linear.

4. Let $f(x)$ be a quadratic polynomial with real coefficients. Prove that $f(x)$ cannot be written as a sum of two periodic functions.

5. Call a real-valued function f *very convex* if

$$\frac{f(x) + f(y)}{2} \geq f\left(\frac{x + y}{2}\right) + |x - y|$$

   holds for all real numbers x and y. Prove that no very convex function exists.